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A thin–tall Boolean algebra which is isomorphic to each of its uncountable subalgebras

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ABSTRACT

Theorem A *($\diamondsuit_{\aleph_1}$). There is a Boolean algebra B with the following properties*:

(1) *B is thin–tall, and*
(2) *B is downward-categorical.*

That is, every uncountable subalgebra of B is isomorphic to B.

The algebra B from Theorem A has some additional properties.
For an ideal K of B, set $\mathrm{cmpl}^B(K) := \{a \in B \mid a \cdot b = 0$ for all $b \in K\}$. We say that K is almost principal if $K \cup \mathrm{cmpl}^B(K)$ generates B.

(3) B is rigid in the following sense. Suppose that I, J are ideals in B and $f : B/I \to B/J$ is a homomorphism with an uncountable range. Then there is an almost principal ideal K of B such that $|\mathrm{cmpl}(K)| \leqslant \aleph_0$, $I \cap K \subseteq J \cap K$, and for every $a \in K$, $f(a/I) = a/J$.
(4) The Stone space of B is sub-Ostaszewski. Boolean-algebraically, this means that: if I is an uncountable ideal in B, then B/I has cardinality $\leqslant \aleph_0$.
(5) Every uncountable subalgebra of B contains an uncountable ideal of B.
(6) Every subset of B consisting of pairwise incomparable elements has cardinality $\leqslant \aleph_0$.
(7) Every uncountable quotient of B has properties (1)–(6).

Assuming $\diamondsuit_{\aleph_1}$ we also construct a Boolean algebra C such that:

(1) C has properties (1) and (4)–(6) from Theorem A, and every uncountable quotient of C has properties (1) and (4)–(6).
(2) C is rigid in the following stronger sense. Suppose that I, J are ideals in C and $f : C/I \to C/J$ is a homomorphism with an uncountable range. Then there is a *principal* ideal K of C such that $|\mathrm{cmpl}(K)| \leqslant \aleph_0$, $I \cap K \subseteq J \cap K$, and for every $a \in K$, $f(a/I) = a/J$.

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1. Introduction

Definition 1.1. A Boolean algebra B is said to be *downward-categorical*, if B is uncountable and every uncountable subalgebra of B is isomorphic to B.

The algebra of finite and cofinite subsets of ω_1 is downward-categorical.

We recall the notion of a thin–tall Boolean algebra (BA). Let X be a *Boolean space*. That is, X is a 0-dimensional compact Hausdorff space. Denote by $\mathrm{Isol}(X)$ the set of isolated points of X, and define $D(X) := X \setminus \mathrm{Isol}(X)$. The Cantor–Bendixon derivatives of X are defined as follows: $D_0(X) = X$, for any ordinal α, $D_{\alpha+1}(X) = D(D_\alpha(X))$, and if δ is a limit ordinal then $D_\delta(X) = \bigcap_{\gamma<\delta} D_\gamma(X)$. We say that X is *thin–tall* if for every $\alpha < \omega_1$, $|\mathrm{Isol}(D_\alpha(X))| = \aleph_0$ and $|\mathrm{Isol}(D_{\omega_1}(X))| < \aleph_0$. A Boolean algebra B is *thin–tall* if its Stone space $\mathrm{Ult}(B)$ (of all ultrafilters of B) is thin–tall.

This work is motivated by the question whether thin–tall downward-categorical Boolean algebras exist. We have the following answer.

Theorem 1.2. (Proved in Theorems 2.7 and 3.1.) *Assume $\diamondsuit_{\aleph_1}$. There is a thin–tall Boolean algebra which is downward-categorical.*

The proof of Theorem 1.2 is divided into two main claims: Theorem 2.7 and Theorem 3.1. In Theorem 2.7 we construct a Boolean algebra which we call a "condensed Boolean algebra". The construction assumes $\diamondsuit_{\aleph_1}$. Condensed BA's are by definition thin–tall, and in Theorem 3.1 we prove that condensed Boolean algebras are downward-categorical.

Condensedness implies several other Boolean algebraic properties. To state these, we need some additional definitions.

A topological space is always nonempty, and in a Boolean algebra B, always $0^B \neq 1^B$. A Boolean space X is *scattered*, if for some ordinal α, $\mathrm{D}_\alpha(X)$ is finite, and it is *unitary*, if for some ordinal α, $\mathrm{D}_\alpha(X)$ is a singleton. We denote the member of this singleton by e^X. The *rank* of X, $\mathrm{rk}(X)$, is the first ordinal α such that $\mathrm{D}_\alpha(X)$ is finite. If $F \subseteq X$ is closed, then $\mathrm{rk}^X(F)$ denotes the rank of F with its induced topology.

A Boolean algebra B is *superatomic*, if its Stone space is scattered, and it is *unitary*, if its Stone space is unitary. Assume that B is superatomic. The *rank* of B, $\mathrm{rk}(B)$, is the rank of $\mathrm{Ult}(B)$. For $a \in B$ and a subset C of B set $C \restriction a := \{c \in C \mid c \leqslant a\}$. If $a \neq 0$, then $B \restriction a$ is a BA, and its rank is denoted by $\mathrm{rk}^B(a)$. Also define, $\mathrm{rk}^B(0^B) := -1$. If $J \subseteq B$ is an ideal, then $\mathrm{rk}(J)$ is defined to be the rank of the BA $J \cup \{-a \mid a \in J\}$. Define $I(B) := \{a \in B \mid \mathrm{rk}^B(a) < \mathrm{rk}(B)\}$. Then $I(B)$ is an ideal in B, and if B is unitary, then $I(B)$ is a maximal ideal.

Let B be a BA and $E \subseteq B$. The *complement ideal* of the set E, $\mathrm{cmpl}^B(E)$, is defined as $\{b \in B \mid \text{for every } e \in E,\ b \cdot e = 0\}$.

Let B be a unitary BA and $I \subseteq B$ be an ideal. We say that I is a *secluded ideal*, if:

(1) I is non-principal.
(2) $\mathrm{rk}(\mathrm{cmpl}(I)) = \mathrm{rk}(B)$. (Hence $I \subseteq I(B)$.)
(3) For every $a \in I(B)$ there are $b \in I$ and $c \in \mathrm{cmpl}(I)$ such that $a = b + c$.

Topologically, this means the following. Let $U_I^B \subseteq \mathrm{Ult}(B)$ be the open set corresponding to I. That is, $U_I^B = \{x \in \mathrm{Ult}(B) \mid x \cap I \neq \emptyset\}$. Then I is secluded iff $\mathrm{cl}(U_I^B) \setminus U_I^B = \{e^{\mathrm{Ult}(B)}\}$ and $\mathrm{rk}(\mathrm{Ult}(B) \setminus U_I^B) = \mathrm{rk}(\mathrm{Ult}(B))$.

Condensed Boolean algebras are rigid in the following strong sense.

Theorem 1.3. (Proved in Corollary 4.4.) *Let B be a condensed Boolean algebra, I, J be ideals of B, and $f : B/I \to B/J$ be a homomorphism. Suppose that $|\mathrm{Rng}(f)| = \aleph_1$. Then there is a countable secluded ideal K of B such that $I \cap \mathrm{cmpl}(K) \subseteq J \cap \mathrm{cmpl}(K)$, and for every $a \in \mathrm{cmpl}(K)$, $f(a/I) = a/J$.*

The rigidity property of Theorem 1.3 is better understood when stated topologically. The following theorem is equivalent to Theorem 1.3. (However, it is not the exact topological translation of Theorem 1.3.) Let $\mathrm{clop}(X)$ denote the Boolean algebra of clopen sets of X.

Theorem 1.3*. *Let X be a Boolean space such that $\mathrm{clop}(X)$ is condensed. Then for every closed subset $Y \subseteq X$ and a continuous function $\varphi : Y \to X$: if $|\mathrm{Rng}(\varphi)| = \aleph_1$, then there is a countable open set V in X such that*:

(1) *The boundary of V is $\{e^X\}$.*
(2) $Y \setminus V \subseteq \mathrm{Rng}(\varphi) \setminus V$.
(3) *For every $x \in Y \setminus V$, $\varphi(x) = x$.*

The following special cases of Theorem 1.3 are more easily understood.

Corollary 1.4. (Proved in Theorem 4.3 and Corollary 4.5.)

(a) *Let B be a condensed Boolean algebra, J be an ideal of B, and $f : B \to B/J$ be a homomorphism. Suppose that $|\mathrm{Rng}(f)| = \aleph_1$. Then there is a countable secluded ideal K of B such that for every $a \in \mathrm{cmpl}(K)$, $f(a) = a/J$.*

(b) *Let B be a condensed Boolean algebra, I be an ideal of B, and $f : B/I \to B$ be a homomorphism. Suppose that $|\mathrm{Rng}(f)| = \aleph_1$. Then there is a countable secluded ideal K of B such that $I \subseteq K$, and for every $a \in \mathrm{cmpl}(K)$, $f(a/I) = a$.*

The conclusion of Corollary 1.4(a) implies the conclusion of Theorem 1.3. In fact, Theorem 1.3* is the topological translation of Corollary 1.4(a) and not of Theorem 1.3. The trivial proof of the above implication appears in the proof of Corollary 4.4.

Condensed Boolean algebras are not retractive. (See Definition 4.6.) In fact, in a condensed Boolean algebra, only trivial ideals have a retract. This fact is stated formally in the following theorem.

Theorem 1.5. (Proved in Corollary 4.7(a).) *Let B be a condensed Boolean algebra, I be an ideal of B. Then the following are equivalent.*

(1) *I has a retract.*
(2) *Either B/I is countable, or there is a countable secluded ideal K of B such that $I \subseteq K$.*

Remark. If I is an ideal in a condensed BA B, then B/I is countable iff I is uncountable. (Corollary 3.8.)

A Boolean algebra B is said to be *quotient-categorical*, if $|B| = \aleph_1$, and every uncountable quotient of B is isomorphic to B.

A condensed BA is not quotient-categorical. In fact, $B/I \cong B$ only if I is trivial. That is:

Corollary 1.6. *Let I be an ideal in a condensed BA B. Then $B/I \cong B$ iff there is a countable secluded ideal K of B such that $I \subseteq K$.*

This corollary is concluded from Corollary 1.4 and Theorem 1.5.

1.1. Sub-Ostaszewski and Ostaszewski algebras

Definition 1.7. (a) A Boolean algebra B is a *sub-Ostaszewski algebra*, if $\mathrm{Ult}(B)$ is thin–tall, and every closed subset of $\mathrm{Ult}(B)$ is either countable or co-countable.

(b) A Boolean algebra B is an *Ostaszewski algebra* if $\mathrm{Ult}(B)$ is thin–tall and unitary, and for every closed subset $F \subseteq \mathrm{Ult}(B)$: if $e^{\mathrm{Ult}(B)}$ is an accumulation point of F, then F is co-countable.

Note that for superatomic BA's the following are equivalent.

(1) Every closed subset of $\mathrm{Ult}(B)$ is either countable or co-countable.
(2) For every uncountable ideal $I \subseteq B$, $|B/I| \leqslant \aleph_0$.

Definition 1.7 is a translation of the topological notions of an Ostaszewski space and a sub-Ostaszewski space to the setting of Boolean algebras and Boolean spaces. The relevant topological definitions can be found in [6].

It follows trivially from the definitions that an Ostaszewski algebra is a sub-Ostaszewski algebra.

A condensed BA is sub-Ostaszewski (Corollary 4.12). Indeed, the following holds.

Proposition 1.8. (Proved in Lemma 3.4 and Corollary 4.12.) *Let B be a condensed algebra and $I \subseteq B$ be an uncountable ideal. Then there is a countable secluded ideal J such that $I \supseteq \mathrm{cmpl}(J)$. Hence B is sub-Ostaszewski.*

However, a condensed algebra is never an Ostaszewski algebra. Every condensed BA is downward-categorical, and we shall next observe that:

Observation 1.9. *(Proved in Corollary 4.7(c).)* A downward-categorical algebra is not an Ostaszewski algebra.

A downward-categorical thin–tall BA must have countable secluded ideals. To see this, let B be a downward-categorical thin–tall BA. Let $a \in B$ be such that $B \restriction a$ is countably infinite and unitary. Let A be the subalgebra of B generated by $I(B \restriction a) \cup I(B \restriction -a)$. Since A is uncountable, $B \cong A$. Also, $I(B \restriction a)$ is a countable secluded ideal in A. So B has a countable secluded ideal.

Let B be a thin–tall BA, $I \subseteq B$ be a countable secluded ideal. Then $e^{\mathrm{Ult}(B)}$ is an accumulation point of U_I^B and $\mathrm{cl}^{\mathrm{Ult}(B)}(U_I^B) = U_I^B \cup \{e^{\mathrm{Ult}(B)}\}$. Hence $|\mathrm{cl}^{\mathrm{Ult}(B)}(U_I^B)| = \aleph_0$. It follows that B is not an Ostaszewski BA.

In [9] (1996) Judith Roitman writes: "under $\diamondsuit$ there is an Ostaszewski space with all three properties", where the three properties are: "(a) retractive; (b) isomorphic to every uncountable image; (c) isomorphic to every uncountable subalgebra". Apparently, Theorem A of [9] is where the above statement is proved. However, Theorem A seems to prove only a restricted version of downward-categoricity. Let B be a thin–tall BA and A be a subalgebra of B. A is a *bounded subalgebra* of B, if for every $b \in I(B)$ there is $a \in A \cap I(B)$ such that for every $c \in A \restriction b$, $c \leqslant a$. The downward-categoricity statement implied by Theorem A seems to be:

- Every uncountable bounded subalgebra B is isomorphic to B.

The following question remains open.

Question 1.10. Is it consistent that there is a thin–tall downward-categorical and quotient-categorical Boolean algebra?

We return to the rigidity property of condensed algebras (Theorem 1.3). A simplification of the construction of a condensed BA yields an Ostaszewski algebra which has a rigidity property slightly stronger than that of Theorem 1.3.

Theorem 1.11. (Proved in Theorem 5.2, Proposition 5.3(b) and Corollary 5.4.)

(a) $(\diamondsuit_{\aleph_1})$ *There is an Ostaszewski algebra B with the following property. Let I, J be ideals of B, and $f : B/I \to B/J$ be a homomorphism. Suppose that $|\mathrm{Rng}(f)| = \aleph_1$. Then there is $b \in I(B)$ such that $I \restriction -b \subseteq J \restriction -b$ and for every $a \in I(B) \restriction -b$, $f(a/I) = a/J$.*

(b) (The topological translation of Part (a).) *There is an Ostaszewski space X such that for every closed subset $Y \subseteq X$ and a continuous function $\varphi : Y \to X$: if $|\mathrm{Rng}(\varphi)| = \aleph_1$, then there is a countable* clopen *set V in X such that*:
  (1) $Y \setminus V \subseteq \mathrm{Rng}(\varphi) \setminus V$.
  (2) *For every $x \in Y \setminus V$, $\varphi(x) = x$.*

Theorem 1.11 is a restatement of Corollary 5.4 which deals with so-called “packed Boolean algebras”.

We observe that a thin–tall downward-categorical BA cannot have the property of Theorem 1.11. Let B be such an algebra and $I \subseteq B$ be a countable secluded ideal. Regard I as a Boolean ring. Then I has an automorphism f such that for every atom a of I, $f(a) \neq a$. Then $f \cup (\mathrm{Id} \restriction \mathrm{cmpl}(I))$ extends to an automorphism g of B, and there is no $b \in I(B)$ such that $g \restriction -b = \mathrm{Id}$.

The results of this work call for the following question.

Question 1.12. Is it consistent that there is a thin–tall downward-categorical BA which is not sub-Ostaszewski?

Theorem 1.2 was found by the authors in 1991. Judith Roitman [10] (2002) proved that the following is consistent: There exists an almost disjoint algebra which is downward-categorical. (An almost disjoint algebra is a subalgebra of $\mathcal{P}(\omega)$ which is generated by the finite subsets of ω and an uncountable almost disjoint family.) The actual result of Roitman is stronger and deals with weak subalgebras.

The analogous question about thin–tall quotient-categorical algebras was investigated by Bonnet and Shelah in [4], and by Roitman in [8] and [9]. Indeed, Bonnet and Shelah [4] constructed with the aid of $\diamondsuit_{\aleph_1}$ a thin–tall retractive quotient-categorical Boolean algebra. Roitman [8] constructed with the aid of CH a thin–tall quotient-categorical algebra.

M. Weese (see Monk and Weese [1991]) proved: If B is downward-categorical, then B is superatomic, and if $2^{\aleph_0} < 2^{\aleph_1}$ holds, then B is thin–tall.

Rigidity theorems for thin–tall Boolean algebras were obtained by M. Weese in [12] and by A. Dow and P. Simon in [5]. Weese, assuming (CH), proved that there is a thin–tall BA B such that for every $f \in \mathrm{Aut}(B)$ there is $\alpha < \omega_1$ such that for every $a \in B$, $a/I_\alpha(B) = f(a)/I_\alpha(B)$. The same rigidity result was obtained by Dow and Simon assuming only ZFC.

2. The construction of a condensed Boolean algebra

The construction of a condensed Boolean algebra is similar to the construction of a strongly concentrated BA in [11].

Let B be a superatomic BA. Set $I_\alpha(B) := \{a \in B \mid \mathrm{rk}^B(a) < \alpha\}$, $\widehat{\mathrm{At}}(B) := \{a \in I(B) \mid B \restriction a$ is unitary$\}$ and for every $\alpha < \mathrm{rk}(B)$ let $\widehat{\mathrm{At}}_\alpha(B) := \{a \in \widehat{\mathrm{At}}(B) \mid \mathrm{rk}(a) = \alpha\}$. Note that if $\alpha \leqslant \mathrm{rk}(B)$, then $I_\alpha(B)$ is an ideal in B.

Let $a, b \in B$. Then $a \sim^B b$ means that either (i) $a, b \neq 0^B$ and $\mathrm{rk}^B(a \triangle b) < \mathrm{rk}^B(a), \mathrm{rk}^B(b)$, or (ii) $a = b = 0^B$. If A is a subset of B, set $-A = \{-a \mid a \in A\}$. An *ideal* I means a proper ideal. That is, $1^B \notin I$. Note that $I \cup -I$ is a subalgebra of B and the isomorphism type of $I \cup -I$ does not depend on B. Recall that $\mathrm{rk}(I) := \mathrm{rk}(I \cup -I)$. For a member $b \in B$ and a subset $A \subseteq B$ set $b \cdot A := \{b \cdot a \mid a \in A\}$.

Suppose that B is a subalgebra of C and $c, d \in C$. Then $[c, d]^B := \{b \in B \mid c \leqslant b \leqslant d\}$. The intervals $(c, d)^B$, $(c, d]^B$ and $[c, d)^B$ are defined in a similar way.

Definition 2.1. Let B be a unitary BA, and I be an ideal in B. I is a *pure ideal of* B, if I is secluded and $\mathrm{rk}^B(a) < \mathrm{rk}(I)$ for all $a \in I$.

The verification of the next proposition is trivial and is left to the reader.

Proposition 2.2. *Let B be a unitary BA and I be an ideal in B.*

(a) *Conditions* (1), (2) *and* (3) *below are equivalent.*
  (1) *I is secluded.*
  (2) *The boundary of both U_I^B and $U_{\mathrm{cmpl}(I)}^B$ is $\{e^{\mathrm{Ult}(B)}\}$, and $\mathrm{rk}(U_{\mathrm{cmpl}(I)}^B \cup \{e^{\mathrm{Ult}(B)}\}) = \mathrm{rk}(\mathrm{Ult}(B))$.*
  (3) *Conditions* (S1)–(S3) *below hold.*
    (S1) *I is non-principal.*
    (S2) *For every $a \in \widehat{\mathrm{At}}(B)$ there is $a' \sim a$ such that either $a' \in I$ or $a' \cdot I = \{0\}$.*
    (S3) *For every $\beta < \mathrm{rk}(B)$ there is an infinite set S of pairwise disjoint elements of $\widehat{\mathrm{At}}(B)$ such that for every $s \in S$, $\mathrm{rk}(s) \geqslant \beta$ and $s \cdot I = \{0\}$.*
    Also, (S2) *is equivalent to the following property: For every $a \in I(B)$, there are $a_1 \in I$ and $a_2 \in \mathrm{cmpl}^B(I)$ such that $a = a_1 + a_2$.*

(b) *If I is secluded, then $I \subseteq I(B)$.*
(c) *If I is secluded, then there is no $b \in I(B)$ such that $I \subseteq B \restriction b$.*
(d) *Let I be a secluded ideal. Then there are a pure ideal J_1 and a principal ideal J_2 such that $J_1 \cap J_2 = \{0\}$, $J_1 \cup J_2$ generates I, and either $\mathrm{rk}(J_2) = \mathrm{rk}(I)$, or $J_2 = \{0\}$.*
(e) *Let I_1 be a secluded ideal or a principal ideal in B, and I_2 be a pure ideal in B. Assume that $\mathrm{rk}(I_1) < \mathrm{rk}(I_2)$. Then the ideal I generated by $I_1 \cup I_2$ is a pure ideal and $\mathrm{rk}(I) = \mathrm{rk}(I_2)$.*

Let B^* be a Boolean algebra, $B \subseteq B^*$ be a subalgebra and $d \in B^*$. Then $B \restriction d$ is an ideal of B. The next definition introduces the notion of a PI-system. Essentially, a PI-system is a pair $\langle B, \mathcal{I} \rangle$, where B is a unitary BA which is either countable or thin–tall, and $\mathcal{I}$ is a family of pure ideals of B. However, it is convenient to capture this situation by introducing another Boolean algebra B^* containing B and a set $D \subseteq B^* \setminus B$ such that the family $\mathcal{I}$ of pure ideals is just $\{B \restriction d \mid d \in D\}$.

Definition 2.3. A *pure ideal system* (*PI-system*) is an object of the form $M = \langle B^\star, B, D \rangle$ where:

(P1) $B^\star$ is an atomic Boolean algebra, B is a subalgebra of $B^\star$, B is unitary, $D \subseteq B^\star \setminus B$, D is infinite, and $B \cup D$ generates $B^\star$.
(P2) $\mathrm{At}(B) = \mathrm{At}(B^\star)$, and $|\mathrm{At}(B)| = \aleph_0$.
(P3) For every $d \in D$, $B \restriction d$ is a pure ideal in B.
(P4) For every distinct $d_1, d_2 \in D$, $d_1 \cdot d_2 \in B$.

$B^\star$, B and D are denoted respectively by $B_M^\star$, B_M and D_M. Define $\mathrm{rk}(M)$ to be $\mathrm{rk}(B)$. Let $I_M^\star$ denote the ideal of $B^\star$ generated by $I(B) \cup D$. (We shall soon verify in Proposition 2.4(b) that $1^{B^\star} \notin I_M^\star$.)

Proposition 2.4. *Let $M = \langle B^\star, B, D \rangle$ be a PI-system.*

(a) *For every $b \in I(B)$, $B^\star \restriction b = B \restriction b$.*
(b) *$I_M^\star$ is a proper ideal of $B_M^\star$, and hence $I_M^\star$ is a maximal ideal of $B_M^\star$.*
(c) *For every $d \in D$, $B \restriction d = I(B) \restriction d$, and $I(B) \restriction d$ is a maximal ideal in the algebra $B^\star \restriction d$.*
(d) *Let $c \in B^\star$ and $b \in B$. If $b \cdot (B \restriction c) = \{0\}$, then $b \cdot c = 0$.*

Proof. (a) Suppose by contradiction that for some $a \in I(B)$, $B^\star \restriction a \neq B \restriction a$. Let $a \in \widehat{\mathrm{At}}(B)$ be of minimal rank such that $B^\star \restriction a \neq B \restriction a$. There is $d \in D$ such that $d \cdot a \notin B$. Since $B \restriction d$ is secluded there is $a' \sim a$ such that either $a' \leqslant d$ or $a' \cdot d = 0$. By the minimality of a, $d \cdot (a - a') \in B$ and $d \cdot (a' - a) \in B$. Suppose that $a' \leqslant d$. Then

$$d \cdot a = d \cdot \bigl(a' - \bigl(a' - a\bigr) + \bigl(a - a'\bigr)\bigr) = d \cdot a' - d \cdot \bigl(a' - a\bigr) + d \cdot \bigl(a - a'\bigr) = a' - d \cdot \bigl(a' - a\bigr) + d \cdot \bigl(a - a'\bigr) \in B.$$

A contradiction.

In the case that $a' \cdot d = 0$, we obtain that $d \cdot a = d \cdot (a - a') \in B$. A contradiction.

(b) Suppose by contradiction that for some $a \in I(B)$ and finite $D_0 \subseteq D$, $a + \sum D_0 = 1$. Let $e \in D \setminus D_0$. Then $e = e \cdot a + \sum\{e \cdot d \mid d \in D_0\}$. By (a) and (P4), $e \in B$. A contradiction.

(c) Let $d \in D$. Suppose that $b \in B \restriction d$. Recall that $B \restriction d$ is secluded. So it follows from Proposition 2.2(b) that $b \in I(B)$. That is, $B \restriction d \subseteq I(B) \restriction d$.

Clearly, $I(B) \restriction d$ is a proper ideal of $B^\star \restriction d$. So it suffices to show that $I(B) \restriction d$ generates $B^\star \restriction d$. Trivially, $\{e \cdot d \mid e \in D \cup I(B)\}$ generates $B^\star \restriction d$. So it suffices to show that $\{e \cdot d \mid e \in D \cup I(B)\} \subseteq I(B)$. For every $e \in D$, $e \cdot d \in B$. From the secludedness of $B \restriction d$ and Proposition 2.2(b) it follows that $e \cdot d \in I(B)$. Suppose next that, $e \in I(B)$, then by (a), $e \cdot d \in I(B)$.

(d) Suppose that $b \cdot c \neq 0$. There is $a \in \mathrm{At}(B^\star)$ such that $a \leqslant b \cdot c$. Since $\mathrm{At}(B^\star) = \mathrm{At}(B)$, it follows that $a \in B$. So $a \in B \restriction c$. So $0 \neq a = b \cdot a \in b \cdot (B \restriction c)$. $\square$

The main property of a condensed BA B is that every uncountable subset of B is somewhere dense. We next define this notion.

Definition 2.5. Let $M = \langle B^\star, B, D\rangle$ be a PI-system.

(a) A *wide interval* of M is a set of the form $[c, -e]^B$, where $c \in I(B)$, $e \in I^\star_M$ and $c \cdot e = 0$. We call $\langle c, e\rangle$ a *wide interval pair* of M. Set $\mathrm{Wip}(M) := \{\langle c, e\rangle \in I(B) \times I^\star_M \mid c \cdot e = 0\}$.

(b) A subset $P \subseteq B$ is *somewhere dense*, if there is $\langle c, e\rangle \in \mathrm{Wip}(M)$ such that for every $\langle c_1, e_1\rangle \in \mathrm{Wip}(M)$: if $[c_1, -e_1]^B \subseteq [c, -e]^B$, then $P \cap [c_1, -e_1]^B \neq \emptyset$.

Note that if $\langle c_1, e_1\rangle, \langle c_2, e_2\rangle \in \mathrm{Wip}(M)$ and $[c_1, -e_1]^B \subseteq [c_2, -e_2]^B$, then $c_2 \leqslant c_1$ and $e_2 \leqslant e_1$. Hence $[c_1, -e_1]^B = [c_2, -e_2]^B$ iff $c_1 = c_2$ and $e_1 = e_2$.

The PI-system that we construct has the property that for every $d \in D$, $B \restriction d$ is countable. Since B is thin–tall, $B \restriction c$ is countable for every $c \in I(B)$. So $[c, -e]^B$ is uncountable for every $\langle c, e\rangle \in \mathrm{Wip}(M)$.

Definition 2.6. A PI-system $M = \langle B^\star, B, D\rangle$ satisfying (C1)–(C4) below is called a *condensed PI-system.*

(C1) B is a thin–tall BA.
(C2) For every $d \in D$, $\mathrm{rk}(B \restriction d) < \omega_1$.
(C3) For every $\alpha < \omega_1$ there is $d \in D$ such that $\mathrm{rk}(B \restriction d) > \alpha$.
(C4) Every uncountable subset P of $I(B)$ is somewhere dense.

A Boolean algebra B is called a *condensed BA* if $B = B_M$ for some condensed PI-system M.

Theorem 2.7 *(Construction Theorem). Assume $\diamondsuit_{\aleph_1}$. There exists a condensed a PI-system.*

Remark 2.8. In the condensed PI-system M which we construct, the following holds.

(C5) For every $\alpha < \omega_1$, $|\{d \in D_M \mid \mathrm{rk}(B_M \restriction d) \leqslant \alpha\}| \leqslant \aleph_0$.

This fact is not used when the various properties of condensed PI-systems are proved, except in Theorem 3.9(a). A condensed PI-system M fulfilling (C5) is called a *narrow condensed* PI-system, and B_M is called a *narrow condensed* BA.

The rest of this section concerns with the proof of Theorem 2.7.

Definition 2.9. (a) Let B be a superatomic BA and $A \subseteq B$. Then $A \sqsubseteq B$, means that for some α, $A = I_\alpha(B) \cup -I_\alpha(B)$.

(b) A PI-system M is countable, if $B^\star_M$ is countable.

(c) Let M and N be PI-systems $M \sqsubseteq N$, if $B_M \sqsubseteq B_N$, $D_M \subseteq D_N$ and for every $d \in D_M$, $B_N \restriction d = B_M \restriction d$.

(d) Let $\{M_i \mid i < \alpha\}$ be a sequence of PI-systems such that for every $i < j < \alpha$, $M_i \sqsubseteq M_j$. We define $M = \bigcup_{i<\alpha} M_i$ to be

$$M := \left\langle \bigcup_{i<\alpha} B^\star_{M_i}, \bigcup_{i<\alpha} B_{M_i}, \bigcup_{i<\alpha} D_i \right\rangle.$$

We shall construct an increasing ω_1-sequence of countable PI-systems $\{M_i \mid i < \omega_1\}$ such that $M = \bigcup_{i<\omega_1} M_i$ will be a condensed PI-system.

Proposition 2.10.

(a) *Let M be a PI-system and $e \in I^\star_M$. There are a finite subset D_0 of D and elements $u, v \in I(B)$ such that $u \leqslant \sum D_0$, $v \cdot \sum D_0 = 0$ and $e = (\sum D_0 - u) + v$. In particular, if $e \in I^\star_M$, then $e \in I(B)$ or $B \restriction e$ is a secluded ideal of B.*
(b) *If $M \sqsubseteq N$, then $B_M = B_N \cap B^\star_M$.*
(c) *If $M \sqsubseteq N$, then $D_M = D_N \cap B^\star_M$.*
(d) *If $M_1 \sqsubseteq M_2 \sqsubseteq M_3$, then $M_1 \sqsubseteq M_3$.*
(e) *Let $\{M_i \mid i < \alpha\}$ and M be as in Definition* 2.9(d) *above. Then M is a PI-system, and for every $i < \alpha$, $M_i \sqsubseteq M$.*

Proof. (a) Suppose that for $i < k$ we have $e_i = (d_i - u_i) + v_i$, where $d_i \in D$, and $u_i, v_i \in I(B)$. Then $\sum_{i<k} e_i = (\sum_{i<k} d_i - u) + v$, where $u \leqslant (\sum_{i<k} u_i) \cdot (\sum_{i<k} d_i)$ and $v = (\sum_{i<k} v_i) - (\sum_{i<k} d_i)$. Recall that $B^\star \restriction a = B \restriction a$ for every $a \in I(B)$. (This was proved in Proposition 2.4(a).) Since $\sum_{i<k} u_i, \sum_{i<k} v_i \in I(B)$, we have that $u, v \in I(B)$. Clearly, $u \leqslant \sum_{i<k} d_i$ and $v \cdot (\sum_{i<k} d_i) = 0$.

Let $e \in I^\star_M$. By the above paragraph it suffices to show that e is the sum of elements of the forms: (1) $d - u$, where $d \in D$ and $u \in I(B)$ and (2) v, where $v \in I(B)$.

For some finite subset $D_0 \subseteq D$ and $a \in I(B)$, $e \leqslant a + \sum D_0$. So $e = e \cdot a + \sum_{d \in D_0} e \cdot d$. Now, $e \cdot a \in I(B)$. So it remains to show that if $e \leqslant d_0$ for some $d_0 \in D$, then e has the desired form. Write e in disjunctive normal form as $e = \sum_{\ell < m} e_\ell$, where each e_ℓ has the form $c \cdot \prod_{j<r} \hat{d}_j$ with $c \in B$ and $\hat{d}_j \in D \cup -D$. We may deal with each e_ℓ separately. Since $e_\ell \leqslant d_0$ we may assume that $\hat{d}_0 = d_0$. We may also assume that the $\hat{d}_j$'s are distinct. If $c \in I(B)$, then $e_\ell \in I(B)$. Suppose next that $c \in -I(B)$. If for some $j > 0$, $\hat{d}_j \in D$, then by (P4), $\hat{d}_0 \cdot \hat{d}_j \in B$. By Proposition 2.4(c), $\hat{d}_0 \cdot \hat{d}_j \in I(B)$ and by Proposition 2.4(a), $e_\ell \in I(B)$. We are left with the case that $\bar{c} := -c \in I(B)$ and for every $j > 0$, $\bar{d}_j := -\hat{d}_j \in D$. We have $e_\ell = d_0 - (\bar{c} + \sum_{j>0} d_0 \cdot \hat{d}_j)$. By (P4) and Proposition 2.4(a), $d_0 \cdot \hat{d}_j \in I(B)$. So $e_\ell = d_0 - u$, where $u \in I(B)$.

(b) Suppose by contradiction that $(B_N \setminus B_M) \cap B^\star_M \neq \emptyset$. Then $(B_N \setminus B_M) \cap I^\star_M \neq \emptyset$. Let $e \in (B_N \setminus B_M) \cap I^\star_M$. Then $e \in I^\star_N$. Write e in the form $(\sum D_0 - u) + v$, where $D_0 \subseteq D_M$ and $u, v \in I(B_M)$. Since $e \notin B_M$, $D_0 \neq \emptyset$.

If $e \in -I(B_N)$, then $1 = e + (-e) \in I^\star_N$. But this contradicts the properness of $I^\star_N$. So $e \in I(B_N)$. Hence $e + u \in I(B_N)$. By Proposition 2.4(a), $B^\star_N \restriction (e+u) = B_N \restriction (e+u)$. Let $d \in D_0$. Then $d \in D_N$ and $d \leqslant e + u$. However, a member of D_N cannot be $\leqslant$ than a member $I(B_N)$. So we obtain a contradiction.

(c) Suppose by contradiction that $d \in (D_N \setminus D_M) \cap B^\star_M$. If $d \in -I^\star_M$, then $1 = d + (-d) \in I^\star_N$, contradicting the properness of $I^\star_N$. So $d = (\sum E_0 - u) + v$, where $E_0 \subseteq D_M$, $u, v \in I(B_M)$, $u < \sum E_0$ and $v \cdot \sum E_0 = 0$. Hence $d \cdot \sum E_0 = \sum E_0 - u \notin B_M$. By (b), $\sum E_0 - u \notin B_N$. So for some $e \in E_0$, $d \cdot e \notin B_N$. But $d \neq e$. So (P4) is contradicted. A contradiction.

(d) and (e) are trivial. □

Definition 2.11. (a) A subset $Q \subseteq \mathrm{Wip}(M)$ is a *dense set of intervals* in M, if for every $\langle c, e\rangle \in \mathrm{Wip}(M)$, there is $\langle c_1, e_1\rangle \in Q$ such that $[c_1, -e_1] \subseteq [c, -e]$.

(b) For $P \subseteq B_M$ let $Q^P_M := \{\langle c, e\rangle \in \mathrm{Wip}(M) \mid P \cap [c, -e] = \emptyset\}$. We say that $P \subseteq B_M$ is *nowhere dense in* M, if Q^P_M is a dense set of intervals in M. Note that P is nowhere dense iff it is not somewhere dense.

(c) Let $M \sqsubseteq N$ be PI-systems and $Q \subseteq \mathrm{Wip}(M)$. N is *convenient for* $\langle Q, M\rangle$ (notation: $\mathrm{Cnvnt}(Q, M; N)$) if Q is a dense set of intervals in M, and for every $a \in I(B_N) \setminus B_M$ there is $\langle c, e\rangle \in Q$ such that $a \in [c, -e]$.

Note that for $Q \subseteq \mathrm{Wip}(M)$, Q is a dense set of intervals in M iff $\mathrm{Cnvnt}(Q, M; M)$ holds. So for $P \subseteq B_M$, P is nowhere dense in M iff $\mathrm{Cnvnt}(Q^P_M, M; M)$ holds.

We construct $\{M_i \mid i < \omega_1\}$ by transfinite induction. We may assume that in a PI-system M, $B^\star_M$ is a subalgebra of $\mathcal{P}(\omega)$ and $\mathrm{At}(B_M) = \mathrm{At}(B^\star_M) = \{\{i\} \mid i \in \omega\}$.

The following lemma will be used in the construction of $M_{\alpha+1}$ from M_α.

Lemma 2.12 *(Successor Case Lemma).*

(a) *Let M be a countable PI-system. Let $\{\langle Q_i, M_i\rangle \mid i \in \omega\}$ be such that for every $i \in \omega$: $M_i \sqsubseteq M$; either $M_i = M$ or $\mathrm{rk}(M_i) < \mathrm{rk}(M)$; and $\mathrm{Cnvnt}(Q_i, M_i; M)$ holds. Then there is a countable PI-system N such that $M \sqsubseteq N$, $\mathrm{rk}(N) = \mathrm{rk}(M) + 1$, $D_N = D_M$, and for every $i \in \omega$, $\mathrm{Cnvnt}(Q_i, M_i; N)$ holds.*

(b) *For a BA B and a subset $C \subseteq B$ we denote by $\mathrm{cl}^B(C)$ the subalgebra of B generated by C. If N is a countable PI-system, then there is $d \subseteq \omega$ such that $N[d] := \langle cl^{\mathcal{P}(\omega)}(B^\star_N \cup \{d\}), B_N, D_N \cup \{d\}\rangle$ is a PI-system, $d \notin B^\star_N$ and $\mathrm{rk}(B_N \restriction d) = \mathrm{rk}(N)$.*

(c) *Let N and d be as in Part* (b)*. Then*:
   (1) *$N \sqsubseteq N[d]$.*
   (2) *For every $\langle Q, M\rangle$: if $\mathrm{Cnvnt}(Q, M; N)$ holds, then $\mathrm{Cnvnt}(Q, M; N[d])$ holds.*

Suppose that δ is a limit ordinal, and for every $i < \delta$, M_i has been constructed. M_δ is defined to be $\bigcup_{i<\delta} M_i$. The following trivial observation assures that the induction hypotheses hold in the limit case.

Proposition 2.13 *(Limit Case Proposition). Let α be a countable limit ordinal and $\{N_i \mid i < \alpha\}$ be a sequence of PI-systems, such that for every $i < j < \alpha$, $N_i \sqsubseteq N_j$. For every $i < \alpha$, let $\{\langle Q_{i,\ell}, M_{i,\ell}\rangle \mid \ell \in L_i\}$ be such that for every $j \geqslant i$ and $\ell \in L_i$, $\mathrm{Cnvnt}(Q_{i,\ell}, M_{i,\ell}; N_j)$. Then for every $i < \alpha$ and $\ell \in L_i$, $\mathrm{Cnvnt}(Q_{i,\ell}, M_{i,\ell}; \bigcup_{j<\alpha} N_j)$.*

Proof. The proof amounts to checking the definitions. □

The central ideas in the proof of the existence of a thin–tall downward-categorical BA appear in the proofs of Theorem 2.7 (Construction Theorem), and Theorem 3.1. The main computational part of the proof appears in the Successor Case Lemma. We start with the computational part. To handle Part (a) of Lemma 2.12, we introduce the following notion of countable forcing.

Definition 2.14. Let $M = \langle B^\star, B, D\rangle$ be a countable PI-system.

(a) We define the partially ordered set $P = P_M$. A member $p \in P$ has the form $\langle \{c_i^p \mid i \in \omega\}, \sigma^p\rangle$, where:

(1) $\{c_i^p \mid i \in \omega\}$ is a pairwise disjoint subset of $I(B)$ such that $\{j \in \omega \mid c_j^p \neq \emptyset\}$ is finite;
(2) σ^p is a finite set of triples of the form $\langle i, d, b\rangle$, where $i \in \omega$, $d \in D$ and $b = c_i^p \cap d$.

Let $p, q \in P$. Then $p \leqslant q$, if for every $i \in \omega$, $c_i^p \subseteq c_i^q$, and $\sigma^p \subseteq \sigma^q$.

(b) Let G be a directed subset of P. For every $i \in \omega$ let $c_i^G = \bigcup\{c_i^p \mid p \in G\}$. Let B_M^G be the subalgebra of $\mathcal{P}(\omega)$ generated by $B \cup \{c_i^G \mid i \in \omega\}$, $(B_M^\star)^G$ be the subalgebra of $\mathcal{P}(\omega)$ generated by $B_M^G \cup D$, and $M^G = \langle (B_M^\star)^G, B_M^G, D\rangle$.

We next define some subsets of P. We shall prove that these subsets are dense in P, and that if G is a directed subset of P meeting all these sets, then M^G is the N required in Part (a) of the “Successor Case Lemma”. In other words, we have a family of tasks. For each task we define a subset of P corresponding to this task. Fulfilling a task is achieved by making sure that G intersects the subset of P which corresponds to this task. Fulfilling all tasks will assure that M^G is as required in Part (a) of the “Successor Case Lemma”.

(c) For $p \in P$ let $c^p = \bigcup_{i\in\omega} c_i^p$. For $b \in I(B)$ define $T_b = T_b^M$ as follows:

$$T_b := \{p \in P \mid b \subseteq c^p\}.$$

Meeting T_b will assure that b is contained in a finite union of c_i^G’s, and that for every $i \in \omega$, $c_i^G \cap b \in B$.

(d) Let $i, k \in \omega$ and $\beta < \mathrm{rk}(M)$. Define $T_{i,k,\beta} = T_{i,k,\beta}^M$ as follows:

$$T_{i,k,\beta} = \bigl\{p \in P \bigm| \text{there is a pairwise disjoint subset } \mathcal{S} \subseteq I(B) \text{ such that } |\mathcal{S}| = k \text{ and for every } a \in \mathcal{S}\text{: } \mathrm{rk}^B(a) \geqslant \beta \text{ and } a \subseteq c_i^p\bigr\}.$$

Meeting all the T_b’s and all $T_{i,k,\beta}$’s will assure that $B_M \sqsubseteq B_M^G$ and that $\mathrm{rk}(B_M^G) = \mathrm{rk}(B_M) + 1$.

(e) For $i \in \omega$ and $d \in D$, define $T_{i,d} = T_{i,d}^M$ as follows:

$$T_{i,d} = \bigl\{p \in P \bigm| \langle i, d, c_i^p \cap d\rangle \in \sigma_p\bigr\}.$$

Meeting all T_b’s, $T_{i,k,\beta}$’s and $T_{i,d}$’s assures that $M \sqsubseteq M^G$.

(f) We define the set of terms $\mathcal{T} = \mathcal{T}_M$. It is the set of all objects t of the form $t = \langle \eta^t, a^t, b^t\rangle$, where η^t is a nonempty finite subset of ω, and a^t, b^t are disjoint members of $I(B)$.

For a directed subset $G \subseteq P$ define $t^G := (\bigcup\{c_i^G \mid i \in \eta^t\} \setminus a^t) \cup b^t$. It will follow that if G meets all T_b’s and all $T_{i,k,\beta}$’s, then $\{t^G \mid t \in \mathcal{T}_M\} = I(B_M^G) \setminus I(B_M)$.

(g) For $t \in \mathcal{T}$ and $p \in P$, we define the *interval of t and p*. It is denoted by $\mathrm{intrvl}(t, p)$ and has the following properties: (i) It is a wide interval; (ii) It is the maximal interval I with the property that for every directed $G \subseteq P$ containing p, $t^G \in I$.

To define $\mathrm{intrvl}(t, p)$, we first define $g^{(t,p)}$ and $h^{(t,p)}$. Once these are defined we set $\mathrm{intrvl}(t, p) := [g^{(t,p)}, -h^{(t,p)}]$. Define

$$g^{(t,p)} := \Bigl(\bigcup\{c_i^p \mid i \in \eta^t\} \setminus a^t\Bigr) \cup b^t.$$

For $i \in \omega$ let $h_i^p := \bigcup\{d \setminus b \mid \langle i, d, b\rangle \in \sigma^p\}$. For $\zeta \subseteq \omega$ let $\hat{c}_\zeta^p := \bigcup\{c_i^p \mid i \in \zeta\}$. For $t \in \mathcal{T}$ and $p \in P$, let

$$h^{(t,p)} := \Bigl(\bigcap\{h_i^p \mid i \in \eta^t\} \cup \hat{c}^p_{\omega\setminus\eta^t} \cup a^t\Bigr) \setminus b^t.$$

Note that $\langle g^{(t,p)}, h^{(t,p)}\rangle \in \mathrm{Wip}(M)$, and if $G \subseteq P$ is directed and $p \in G$, then $t^G \in \mathrm{intrvl}(t, p)$.

(h) Suppose $L \sqsubseteq M$ and $Q \subseteq \mathrm{Wip}(L)$ fulfill $\mathrm{Cnvnt}(Q, L; M)$ and that either $L = M$ or $\mathrm{rk}(L) < \mathrm{rk}(M)$. For $t \in \mathcal{T}_M$ define $T_{t,Q} = T_{t,Q}^M$ as follows:

$$T_{t,Q} = \bigl\{p \in P \bigm| \text{there is } \langle c, e\rangle \in Q \text{ such that } \mathrm{intrvl}(t, p) \subseteq [c, -e]\bigr\}.$$

Meeting $T_{t,Q}$ will assure that t^G lies in an interval belonging to Q.

The proof of the next two lemmas is technical but straightforward. We skip their proof.

Part (b) of the following lemma will serve as a convenient intermediate step in the proof of the lemma that follows it.

Lemma 2.15 *(Interval Lemma). Let M be a countable PI-system, $t \in \mathcal{T}_M$ and $p \in P_M$. Set $P = P_M$.*

(a) *Let $G \subseteq P$ be directed and $p \in G$. Then $t^G \in \mathrm{intrvl}(t, p)$. If $p \geqslant q$, then $\mathrm{intrvl}(t, p) \subseteq \mathrm{intrvl}(t, q)$.*
(b) *Let $\langle g, h\rangle \in \mathrm{Wip}(M)$ be such that $g^{(t,p)} \subseteq g$ and $h^{(t,p)} \subseteq h$. Then there is $q \geqslant p$ such that $\mathrm{intrvl}(t, q) \subseteq [g, -h]$.*

Lemma 2.16 *(Density Lemma). Let $M = \langle B^\star, B, D\rangle$ be a countable PI-system, $P = P_M$ and $\boldsymbol{\tau} = \boldsymbol{\tau}_M$.*

(a) *For every $b \in I(B)$, T^M_b is dense in P.*
(b) *For every $i, k \in \omega$ and $\beta < \mathrm{rk}(M)$, $T^M_{i,k,\beta}$ is dense in P.*
(c) *For every $i \in \omega$ and $d \in D$, $T^M_{i,d}$ is dense in P.*
(d) *Let L, Q be as in Definition 2.14(h). Then for every $t \in \boldsymbol{\tau}$, $T^M_{t,Q}$ is dense in P.*

Proof of Lemma 2.12 (Successor Case Lemma). (a) Let M and $\{\langle Q_i, M_i\rangle \mid i \in \omega\}$ be as in Lemma 2.12(a). Let G be a directed subset of P_M such that G intersects all the dense subsets of P defined in Definition 2.14(c), (d), (e) and (h). That is,

(1) $G \cap T^M_b \neq \emptyset$ for every $b \in I(B_M)$;
(2) $G \cap T^M_{i,k,\beta} \neq \emptyset$ for every $i, k \in \omega$ and $\beta < \mathrm{rk}(M)$;
(3) $G \cap T^M_{i,d} \neq \emptyset$ for every $i \in \omega$ and $d \in D_M$;
(4) $G \cap T^M_{t,Q_j} \neq \emptyset$ for every $t \in \boldsymbol{\tau}_M$ and $j \in \omega$.

Let N be defined as follows: $B_N := \mathrm{cl}^{\mathcal{P}(\omega)}(B_M \cup \{c^G_i \mid i \in \omega\})$, $D_N := D_M$ and $B^\star_N := cl^{\mathcal{P}(\omega)}(B_N \cup D_N)$. We check that N is as required. At first we show that N is a PI-system. In fact, we show that if G intersects every dense set of the forms T^M_b, $T^M_{i,k,\beta}$ and $T^M_{i,d}$, then N is a PI-system.

The following facts imply that B_N is superatomic: (1) B_N is generated by $B_M \cup \{c^G_i \mid i \in \omega\}$, (2) B_M is superatomic, (3) $\{c^G_i \mid i \in \omega\}$ is a partition of ω.

Since $\{\{n\} \mid n \in \omega\} \subseteq B_N \subseteq B^\star_N \subseteq \mathcal{P}(\omega)$, it follows that $B^\star_N$ is atomic and that $\mathrm{At}(B_N) = \mathrm{At}(B^\star_N)$ and $|\mathrm{At}(B_N)| = \aleph_0$. So N fulfills (P2).

Since M is a PI-system, $B_M \subseteq B_N$ and $D_N = D_M$, it follows that for every $d_1, d_2 \in D_N$, $d_1 \cap d_2 \in B_N$. So N fulfills (P4).

Fact 1: For every $b \in I(B_M)$, $B_N \restriction b = B_M \restriction b$. Let $b \in I(B_M)$. Choose $p \in G \cap T_b$. Then for every $i \in \omega$, $b \cap c^G_i = c^p_i \cap b \in I(B_M)$. Hence $B_N \restriction b = B_M \restriction b$. $\square$

Fact 2: For every $i \in \omega$ and $a \in B_N \restriction c^G_i$, either $a \in I(B_M)$ or $c^G_i \setminus a \in I(B_M)$. Let $a \in B_N \restriction c^G_i$. Then either (i) there is $b \in I(B_M)$ such that $a = c^G_i \cap b$ or (ii) there is $b \in I(B_M)$ such that $a = c^G_i \setminus b$. This is so since for every $j \neq i$, $c^G_j \cap c^G_i = \emptyset$ and since $I(B_M)$ is a maximal ideal of B_M.

By Fact 1, if (i) happens then $a \in I(B_M)$, and if (ii) happens, then $c^G_i \setminus a \in I(B_M)$. $\square$

Fact 3: For every $i \in \omega$, $\mathrm{rk}(B_M \restriction c^G_i) = \mathrm{rk}(B_M)$. Since $B_M \restriction c^G_i$ is an ideal in B_M, $\mathrm{rk}(B_M \restriction c^G_i) \leqslant \mathrm{rk}(B_M)$. Let $i \in \omega$, $\beta < \mathrm{rk}(B_M)$ and $k \in \omega$. Choose $p \in G \cap T_{i,k,\beta}$. Let $\mathcal{S} \subseteq I(B)$ be a pairwise disjoint set such that $|\mathcal{S}| = k$ and for every $a \in \mathcal{S}$: $\mathrm{rk}^{B_M}(a) \geqslant \beta$ and $a \subseteq c^p_i$. So $\mathrm{rk}(B_M \restriction c^G_i) > \beta$. We conclude that $\mathrm{rk}(B_M \restriction c^G_i) \geqslant \sup\{\beta + 1 \mid \beta < \mathrm{rk}(B_M)\} = \mathrm{rk}(B_M)$. So $\mathrm{rk}(B_M \restriction c^G_i) = \mathrm{rk}(B_M)$. $\square$

Fact 4: For every $i \in \omega$, $c^G_i \notin B_M$. Suppose by contradiction that $c^G_i \in B_M$. By Fact 3, $\mathrm{rk}^{B_M}(c^G_i) = \mathrm{rk}(B_M \restriction c^G_i) = \mathrm{rk}(B_M)$. Let $j \neq i$. Then $\mathrm{rk}^{B_M}(-c^G_i) = \mathrm{rk}(B_M \restriction -c^G_i) \geqslant \mathrm{rk}(B_M \restriction c^G_j) = \mathrm{rk}(B_M)$. This implies that B_M is not unitary. A contradiction. So $c^G_i \notin B_M$. $\square$

Fact 5: For every $i \in \omega$, $\mathrm{rk}^{B_N}(c^G_i) = \mathrm{rk}(B_M)$. By Facts 2 and 4, $B_N \restriction c^G_i \cong (B_M \restriction c^G_i) \cup -(B_M \restriction c^G_i)$. So

$$\mathrm{rk}^{B_N}\big(c^G_i\big) = \mathrm{rk}\big(B_N \restriction c^G_i\big) = \mathrm{rk}\big(B_M \restriction c^G_i\big) = \mathrm{rk}(B_M). \qquad \square$$

Fact 6: $\mathrm{rk}(B_N) = \mathrm{rk}(B_M) + 1$ and B_N is unitary. For every $a \in B_N$ there is a finite set $\rho \subseteq \omega$ such that $a \subseteq \bigcup_{i\in\rho} c^G_i$ or $-a \subseteq \bigcup_{i\in\rho} c^G_i$. Since for every $i \in \omega$, $\mathrm{rk}^{B_N}(c^G_i) = \mathrm{rk}(B_M)$, it follows that $\mathrm{rk}(B_N) = \mathrm{rk}(B_M) + 1$ and B_N is unitary. $\square$

Clause (P1) in the definition of a PI-system consisted of the following items:

(P1.1) $B^\star$ is an atomic Boolean algebra,
(P1.2) B is a subalgebra of $B^\star$, B is unitary,
(P1.3) $D \subseteq B^\star \setminus B$, D is infinite,
(P1.4) $B \cup D$ generates $B^\star$.

We already know that N satisfies (P1.1), (P1.2) and (P1.4) and that D_N is infinite. It remains to check that $D_N \subseteq B^\star_N \setminus B_N$.

Fact 7: For every $d \in D_N$ and $i \in \omega$, $c^G_i \cap d \in I(B_M)$. Let $d \in D_N$ and $i \in \omega$. Then $d \in D_M$. Let $p \in G \cap T_{i,d}$. So $\langle i, d, d \cap c^p_i\rangle \in \sigma_p$. So for every $q \in G$: if $q \geqslant p$, then $c^q_i \cap d = d \cap c^p_i$. Hence $c^G_i \cap d = d \cap c^p_i \in I(B_M)$. The fact that $d \cap c^p_i \in I(B_M)$ follows from Proposition 2.4(a). $\square$

Fact 8: Every $e \in B_N$ has the one of following forms:

(I) $((\bigcup_{i\in\zeta} c^G_i) \setminus a) \cup b \cup ((\bigcap_{i\in\rho} -c^G_i) \setminus c)$, where ρ, ζ are finite subsets of ω and $a, b, c \in I(B_M)$;
(II) $((\bigcup_{i\in\zeta} c^G_i) \setminus a) \cup b$, where ζ is a finite subset of ω and $a, b \in I(B_M)$.

The proof of Fact 8 uses the following facts: (i) Fact 1, (ii) $\{c_i^G \mid i \in \omega\}$ is a partition of ω. The rest of the verification is left to the reader. □

Fact 9: Let $d \in D_N$. Then there are no ζ, b such that ζ is a finite subset of ω, $b \in I(B_M)$ and $d \subseteq (\bigcup_{i\in\zeta} c_i^G) \cup b$. Suppose otherwise. Then $d = (d \cap (\bigcup_{i\in\zeta} c_i^G)) \cup (d \cap b) \in I(B_M) \subseteq B_M$. A contradiction. □

Fact 10: Let $d \in D_N$. Then there are no ρ, c such that ρ is a finite subset of ω, $c \in I(B_M)$ and $(\bigcap_{i\in\rho} -c_i^G) \setminus c \subseteq d$. Suppose otherwise. Since G intersects T_c, $\xi := \{i \in \omega \mid c_i^G \cap c \neq \emptyset\}$ is finite. Let $j \in \omega \setminus (\rho \cup \xi)$. Then $c_j^G \subseteq d$. So by Fact 7, $c_j^G \in I(B_M) \subseteq B_M$. This contradicts Fact 4. □

Fact 11: $D_N \subseteq B_N^\star \setminus B_N$. Suppose by contradiction that $d \in D_N \cap B_N$. Then d has either form (I) or form (II) of Fact 8. If the former happens, then $(\bigcap_{i\in\rho} -c_i^G) \setminus c \subseteq d$, where ρ is a finite subset of ω and $c \in I(B_M)$. This contradicts Fact 10. If the latter happens, then $d \subseteq (\bigcup_{i\in\zeta} c_i^G) \cup b$, where ζ is a finite subset of ω and $b \in I(B_M)$. This contradicts Fact 9. □

We have shown that N fulfills Clause (P1) in the definition of a PI-system.

We now show that N fulfills Clause (P3) in the definition of a PI-system. That is, for every $d \in D_N$, $B_N \restriction d$ is a pure ideal. We show that $B_N \restriction d$ fulfills Clauses (S1)–(S3) of Proposition 2.2.

By Fact 11, $d \notin B_N$. So $B_N \restriction d$ is non-principal. So (S1) of Proposition 2.2 holds.

Fact 12: $B_N \restriction d = I(B_M) \restriction d$. Let $e \in B_N \restriction d$. By Fact 10, e does not have form (I) of Fact 8. So e has form (II). Then by Fact 7, $e \in I(B_M)$. So $B_N \restriction d \subseteq I(B_M) \restriction d$. Clearly, $I(B_M) \restriction d \subseteq B_N \restriction d$. So $I(B_M) \restriction d = B_N \restriction d$. □

Since $I(B_M) \restriction d$ is a pure ideal in B_M, it follows that for every $c \in B_N \restriction d$, $\mathrm{rk}^{B_N}(c) = \mathrm{rk}^{B_M}(c) < \mathrm{rk}(B_N \restriction d)$. That is, the last clause in the definition of pureness holds.

Fact 13: Let $a \in \widehat{\mathrm{At}}(B_N)$. Then either $a \in \widehat{\mathrm{At}}(B_M)$, or for some $i \in \omega$ and $b, c \in I(B_M)$, $a = (c_i^G \setminus b) \cup c$. The proof of this fact is left to the reader. □

We prove that $B_N \restriction d$ fulfills Clause (S2) of Proposition 2.2(a). Let $a \in \widehat{\mathrm{At}}(B_N)$. If the first case of Fact 13 happens, then applying Proposition 2.4(a) to M, we conclude that $a \setminus d, a \cap d \in B_M$. So either $a \setminus d \sim^{B_N} a$ or $a \cap d \sim^{B_N} a$. If the second case of Fact 13 happens, then $a \setminus d \sim^{B_N} a$. In both cases we found $a' \sim^{B_N} a$ such that either $a' \cdot (B_N \restriction d) = \{0\}$ or $a' \in B_N \restriction d$. That is, $B_N \restriction d$ fulfills Clause (S2).

We prove that $B_N \restriction d$ fulfills (S3). Let $\beta < \mathrm{rk}(B_N)$. Since $\mathrm{rk}(B_N) = \mathrm{rk}(B_M) + 1$, we may assume that $\beta = \mathrm{rk}(B_M)$. For every $i \in \omega$, $c_i^G \cap d \in I(B_M)$. So $\mathrm{rk}^{B_N}(c_i^G \cap d) < \mathrm{rk}(B_M) = \mathrm{rk}^{B_N}(c_i^G)$. It follows that $\mathrm{rk}^{B_N}(c_i^G \setminus d) = \beta$. Let $S = \{c_i^G \setminus d \mid i \in \omega\}$. Then S is a pairwise disjoint set, and for every $s \in S$, $(B_N \restriction d) \cdot s = \{0\}$. So $B_N \restriction d$ fulfills (S3).

We have shown that $B_N \restriction d$ is a pure ideal. This concludes the proof that N is a PI-system.

We check that $M \sqsubseteq N$. By Fact 1, $I(B_M) \subseteq I_{\mathrm{rk}(M)}(B_N)$. Let $a \in I(B_N)$. By Fact 13, there are ρ, b, c such that $\rho \subseteq \omega$ is finite, $b, c \in I(B_M)$ and $a = ((\bigcup_{i\in\rho} c_i^G) \setminus b) \cup c$. Suppose that $\mathrm{rk}^N(a) < \mathrm{rk}(B_M)$. Then by Fact 6, $\rho = \emptyset$. So $a \in I(B_M)$. We have shown that $I_{\mathrm{rk}(M)}(B_N) \subseteq I(B_M)$. Hence $I_{\mathrm{rk}(M)}(B_N) = I(B_M)$. So

$$B_M = I_{\mathrm{rk}(M)}(B_M) \cup -I_{\mathrm{rk}(M)}(B_M) = I_{\mathrm{rk}(M)}(B_N) \cup -I_{\mathrm{rk}(M)}(B_N).$$

That is, $B_M \sqsubseteq B_N$.

By definition, $D_N = D_M$. Let $d \in D_N$. By Fact 12, $B_N \restriction d = I(B_M) \restriction d$. So $B_N \restriction d = B_M \restriction d$. We have shown that $M \sqsubseteq N$.

Lastly, we show that for every $i \in \omega$, $\mathrm{Cnvnt}(Q_i, M_i; N)$ holds. Let $a \in I(B_N)$. There is $t \in \mathcal{T}_N$ such that $a = t^G$. Let $p \in G \cap T_{t,Q_i}$. So there is $\langle c, e\rangle \in Q_i$ such that $\mathrm{Intrvl}(t, p) \subseteq [c, -e]$. So $a = t^G \in \mathrm{Intrvl}(t, p) \subseteq [c, -e]$. This shows that $\mathrm{Cnvnt}(Q_i, M_i; N)$ holds.

(b) Let N be as in Part (b). Let G be a directed subset of P_N which intersects every dense set of the forms (i) T_b^N, $b \in I(B)$; (ii) $T_{i,k,\beta}^N$, $i, k \in \omega$, $\beta < \mathrm{rk}(N)$; and (iii) $T_{i,d'}^N$, $i \in \omega$, $d' \in D_N$. Let $d := c_0^G$.

We show that $N[d]$ is a PI-system. The verification of (P1) and (P2) is trivial. We show that $B_N \restriction c_0^G$ is a pure ideal. In Part (a) Fact 3 we showed that $\mathrm{rk}(B_N \restriction c_0^G) = \mathrm{rk}(B_N)$. We also showed that $B_N \restriction c_0^G \subseteq I(B_N)$. This implies that for every $a \in B_N \restriction c_0^G$, $\mathrm{rk}^{B_N}(a) < \mathrm{rk}(B_N \restriction c_0^G)$. In particular, $B_N \restriction c_0^G$ is non-principal. So (S1) of Proposition 2.2 holds.

Let $a \in \widehat{\mathrm{At}}(B_N)$. The fact that G intersects T_a implies that $a \cap c_0^G \in B_N$. So either $a \cap c_0^G \sim^{B_N} a$ and $a \cap c_0^G \subseteq c_0^G$ or $a \setminus c_0^G \sim^{B_N} a$ and $(a \setminus c_0^G) \cap c_0^G = \emptyset$. This means that (S2) of Proposition 2.2 holds.

Let $\beta < \mathrm{rk}(B_N)$. Then for every $i \in \omega$, $(B_N \restriction c_i^G) \cap \widehat{\mathrm{At}}_\beta(B_N) \neq \emptyset$. Since the c_i^G's are pairwise disjoint, it follows that $B_N \restriction c_0^G$ fulfills (S3) of Proposition 2.2. Hence $B_N \restriction c_0^G$ is a secluded ideal. By the above paragraph, $B_N \restriction c_0^G$ is also pure. We have shown that $N[d]$ fulfills (P3) in the definition of a PI-system.

Let $e \in D_N$. Choose $p \in G \cap T_{0,e}$ and let $\langle 0, e, b\rangle \in \sigma^p$. Then $c_0^G \cap e = b \in B_N$. Hence (P4) is fulfilled. This proves (b).

(c) Part (c) holds almost by definition. The only fact that needs verification is that $c_0^G \notin B_N$. This has been already checked. □

Proof of Theorem 2.7 (Construction Theorem). For a well-ordered set L, $\mathrm{Lim}(L)$ denotes the set of limit points of L. So $\mathrm{Lim}(\omega_1) = \{\omega \cdot \alpha \mid 0 \neq \alpha \in \omega_1\}$. We define by induction on $\alpha \in \mathrm{Lim}(\omega_1)$ the following objects:

(1) A countable PI-system M_α such that $B_{M_\alpha}^*$ is a subalgebra of $\mathcal{P}(\omega)$ and $\mathrm{At}(B_{M_\alpha}^*) = \{\{n\} \mid n \in \omega\}$.
(2) A dense set of intervals Q_α of M_α.
(3) A bijection $f_\alpha : \alpha \to B_{M_\alpha}^\star$.

Let $\{S_\alpha \mid \alpha < \omega_1\}$ be a $\diamondsuit$-sequence for ω_1.

The induction hypotheses are as follows. Suppose that $\gamma \in \mathrm{Lim}(\omega_1)$ and M_α, Q_α, f_α have been defined for every $\alpha \in \mathrm{Lim}(\gamma)$. Then for every $\alpha, \beta \in \mathrm{Lim}(\gamma)$ such that $\alpha \leqslant \beta$:

(H1) If $\alpha = \omega \cdot \alpha'$, then $\mathrm{rk}(M_\alpha) = \alpha'$.
(H2) $M_\alpha \sqsubseteq M_\beta$.
(H3) $\mathrm{Cnvnt}(Q_\alpha, M_\alpha; M_\beta)$ holds.
(H4) $f_\alpha \subseteq f_\beta$.

The case $\gamma = \omega$: Take D_ω to be any infinite partition of ω into infinite sets. Let $B^\star_\omega$ be the subalgebra of $\mathcal{P}(\omega)$ generated by $\{\{n\} \mid n \in \omega\} \cup D_\omega$, B_ω be the algebra of finite and cofinite subsets of ω and $M_\omega = \langle B^\star_\omega, B_\omega, D_\omega\rangle$. Set $Q_\omega = \mathrm{Wip}(M_\omega)$ and let f_ω be a bijection between ω and $B^\star_\omega$. It is obvious that the induction hypotheses hold.

The limit case: Let $\gamma \in \mathrm{Lim}(\mathrm{Lim}(\omega_1))$, and suppose that for every $\alpha \in \mathrm{Lim}(\gamma)$, M_α, Q_α and f_α have been defined. Let $M_\gamma = \bigcup_{\alpha \in \mathrm{Lim}(\gamma)} M_\alpha$ and $f_\gamma = \bigcup_{\alpha \in \mathrm{Lim}(\gamma)} f_\alpha$.

Let $P_\gamma := f_\gamma(S_\gamma)$. If $P_\gamma \subseteq I(B_{M_\gamma})$, and is nowhere dense in M_γ, let $Q_\gamma := Q^{P_\gamma}_{M_\gamma}$, that is,

$$Q_\gamma = \bigl\{\langle c, e\rangle \in \mathrm{Wip}(M_\gamma) \bigm| P_\gamma \cap [c, -e] = \emptyset\bigr\}.$$

Otherwise, let $Q_\gamma := \mathrm{Wip}(M_\gamma)$.

It follows from the Limit Case Proposition 2.13 that M_α, Q_α, f_α, $\alpha \in \mathrm{Lim}(\gamma + \omega)$ satisfy the induction hypotheses.

The successor case: Let γ be a successor in $\mathrm{Lim}(\omega_1)$, and suppose that M_β, Q_β and f_β have been defined for every $\beta \in \mathrm{Lim}(\gamma)$. Let $\alpha \in \mathrm{Lim}(\omega_1)$ be such that $\gamma = \alpha + \omega$. By Part (a) of the Successor Case Lemma 2.12, there is a countable $M'_\gamma \sqsupseteq M_\alpha$ such that $\mathrm{rk}(M'_\gamma) = \mathrm{rk}(M_\alpha) + 1$, $D_{M'_\gamma} = D_{M_\alpha}$, and for every limit $\beta \in \mathrm{Lim}(\gamma)$, $\mathrm{Cnvnt}(Q_\beta, M_\beta; M'_\gamma)$ holds.

Let $d_\gamma \subseteq \omega$ and $M'_\gamma[d_\gamma]$ be as assured by Part (b) of the Successor Case Lemma 2.12 applied to M'_γ, and let $M_\gamma := M'_\gamma[d_\gamma]$.

We now define f_γ and Q_γ. Let f_γ be a bijection between γ and $B^\star_{M_\gamma}$ such that $f_\gamma \supseteq f_\alpha$. Let $P_\gamma := f_\gamma(S_\gamma)$. If $P_\gamma \subseteq I(B_{M_\gamma})$, and is nowhere dense in M_γ, let $Q_\gamma := Q^{P_\gamma}_{M_\gamma}$. Otherwise, let $Q_\gamma := \mathrm{Wip}(M_\gamma)$.

It follows from the three parts of the Successor Case Lemma 2.12 and from Proposition 2.10(d) that the induction hypotheses hold.

Let $M = \bigcup_{\alpha < \omega_1} M_\alpha$. We show that M satisfies (C1)–(C4) of Theorem 2.7.

(C1) By Proposition 2.10(e), for every $\alpha \in \mathrm{Lim}(\omega_1)$, $M_\alpha \sqsubseteq M$. Since for every $\alpha = \omega \cdot \alpha' \in \mathrm{Lim}(\omega_1)$, M_α is countable and $\mathrm{rk}(M_\alpha) = \alpha'$, we conclude that $\mathrm{rk}(M) = \omega_1$ and B_M is a thin–tall BA. Hence (C1) holds.

(C2) Let $d \in D_M$. Hence for some $\alpha \in \mathrm{Lim}(\omega_1)$, $d \in D_{M_\alpha}$. Since $M_\alpha \sqsubseteq M$, $B_M \restriction d = B_{M_\alpha} \restriction d$. Hence $\mathrm{rk}(B_M \restriction d) = \mathrm{rk}(B_{M_\alpha} \restriction d) \leqslant \alpha$. So (C2) holds.

(C3) Let $\alpha \in \mathrm{Lim}(\omega_1)$ be a successor element of $\mathrm{Lim}(\omega_1)$. By our construction, d_α is defined and $\mathrm{rk}(B_M \restriction d_\alpha) = \mathrm{rk}(M_\alpha \restriction d_\alpha) = \alpha'$. So (C3) holds.

(C4) holds. Suppose by contradiction that $P \subseteq I(B_M)$ is uncountable and nowhere dense. Let $N := \langle B^\star_M, B_M, D_M, I(B_M), I^\star_M, P\rangle$. That is, N is the expansion of $B^\star_M$, obtained by adding unary predicates which represent B_M, D_M, $I(B_M)$, $I^\star_M$ and P (so the universe of N is $B^\star_M$, and for instance, $N \models P(x)$ if and only if $x \in P$). For $\alpha \in \mathrm{Lim}(\omega_1)$ let N_α be the substructure of N whose universe is $B^\star_{M_\alpha}$. Then $C := \{\alpha \in \mathrm{Lim}(\omega_1) \mid N_\alpha \prec N\}$ is a closed unbounded subset of ω_1. Let $f = \bigcup\{f_\alpha \mid \alpha \in \mathrm{Lim}(\omega_1)\}$. By $\diamondsuit_{\aleph_1}$, $S := \{\alpha \in \omega_1 \mid f^{-1}(P) \cap \alpha = S_\alpha\}$ is stationary in ω_1.

Denote the universe of any structure N' by $|N'|$. Let $\beta \in S \cap C$. The fact that P is nowhere dense in M is expressible by a first order sentence in the language of N. Since $N_\beta \prec N$, the same sentence holds in N_β, so $P \cap |N_\beta|$ is nowhere dense in M_β. Since $\beta \in S$, $f^{-1}(P) \cap \beta = S_\beta$. So $f_\beta(S_\beta) = f(S_\beta) = P \cap f(\beta) = P \cap |N_\beta|$. That is, $f_\beta(S_\beta)$ is nowhere dense in M_β. So $P_\beta = f(S_\beta)$ and $Q_\beta = Q^{P_\beta}_{M_\beta}$. Let $a \in P \setminus |N_\beta|$. Since $a \in I(B_M) \setminus B_{M_\beta}$ and $\mathrm{Cnvnt}(Q_\beta, M_\beta; M)$ holds, there is $\langle c, e\rangle \in Q_\beta$ such that $a \in [c, -e]$. By the definition of Q_β, $P_\beta \cap [c, -e] = \emptyset$. So $N_\beta \models \neg\exists x(P(x) \wedge (x \in [c, -e]))$, whereas $N \models \exists x(P(x) \wedge (x \in [c, -e]))$. (This is so, since $P(a) \wedge (a \in [c, -e])$ holds.) This contradicts the fact that $N_\beta \prec N$. So M satisfies (C4). $\square$

We conclude this section by observing that there are $2^{\aleph_1}$ pairwise non-isomorphic condensed BA's. Let B, C be uncountable Boolean algebras. We say that B and C are *far*, if there is no uncountable BA which is embeddable in both B and C. Note that since condensed BA's are downward-categorical, being non-isomorphic implies being far.

We may also define the notion of quotient-far. B and C are *quotient-far*, if they do not have isomorphic uncountable quotients.

We prove a little more. Namely, there is a family $\mathcal{B}$ of condensed Boolean algebras such that $|\mathcal{B}| = 2^{\aleph_1}$ and for every distinct $B, C \in \mathcal{B}$, B and C are far and quotient-far.

Let $B \simeq C$ mean that there are countable ideals I, J of B and C such that $B/I \cong C/J$. Note that $\simeq$ is an equivalence relation on the class of Boolean algebras and that for sub-Ostaszewski Boolean algebras, $B \simeq C$ iff there are ideals I, J of B and C such that B/I is uncountable and $B/I \cong C/J$.

Theorem 2.17. ($\diamondsuit_{\aleph_1}$) *There is a set $\mathcal{B}$ of condensed BA's such that $|\mathcal{B}| = 2^{\aleph_1}$, and for every distinct $B_1, B_2 \in \mathcal{B}$, $B_1 \not\simeq B_2$.*

Proof. Let $T = \{\tau \mid \text{there is } \alpha \in \omega_1 \text{ such that } \tau : \alpha \to \{0,1\}\}$, $T^{(<\beta)} = \{\tau \in T \mid \mathrm{Dom}(\tau) < \beta\}$, $T^{(\beta)} = \{\tau \in T \mid \mathrm{Dom}(\tau) = \beta\}$ and $\widehat{T} = \{0,1\}^{\omega_1}$. We shall construct a family of countable PI-systems $\{M_\tau \mid \tau \in T\}$. The construction takes care that for every $\sigma \in \widehat{T}$, $M_\sigma := \bigcup_{\alpha \in \omega_1} M_{\sigma \restriction \alpha}$ is well defined and is a condensed PI-system. For $\sigma \in T \cup \widehat{T}$ denote B_{M_σ} by B_σ. Then B_σ is made to be a subalgebra of $\mathcal{P}(\omega)$ and $\mathrm{At}(B_\sigma) = \{\{n\} \mid n \in \omega\}$.

Assume temporarily that the construction has the property that for every distinct $\sigma, \nu \in \widehat{T}$, $B_\sigma \neq B_\nu$. Suppose that $f : B_\sigma \cong B_\nu$. Then there is $\pi_f \in \mathrm{Sym}(\omega)$ such that $f(a) = \pi_f[a]$ for every $a \in B_\sigma$. This implies that for every $\sigma \in \widehat{T}$, $|\{\nu \in \widehat{T} \mid B_\nu \cong B_\sigma\}| \leqslant 2^{\aleph_0} = \aleph_1$. It follows that there is a subset $\mathcal{B} \subseteq \{M_\sigma \mid \sigma \in \widehat{T}\}$, such that $|\mathcal{B}| = 2^{\aleph_1}$, and for every distinct $B_1, B_2 \in \mathcal{B}$, $B_1 \not\cong B_2$.

We have to do more in order to obtain that for every distinct $\sigma, \nu \in \widehat{T}$, $B_\sigma \not\simeq B_\nu$. This can be done assuming CH, but since we have to use $\diamondsuit$ elsewhere in the proof, we also use $\diamondsuit$ here.

We now construct $\{M_\sigma \mid \sigma \in T\}$. In fact, for every $\sigma \in T$ we shall have the following objects:

(1) A countable PI-system M_σ. Set $B_\sigma := B_{M_\sigma}$ and $B^*_\sigma := B^*_{M_\sigma}$. Then B_σ is a subalgebra of $\mathcal{P}(\omega)$ and $\mathrm{At}(B_\sigma) = \{\{n\} \mid n \in \omega\}$.
(2) A dense set of intervals Q_σ of M_σ.
(3) A bijection $f_\sigma : \omega \cdot \alpha \to B^*_\sigma$, where $\alpha = \mathrm{Dom}(\sigma)$.

Let $\alpha \in \omega_1$ and suppose that M_σ has been constructed for every $\sigma \in T^{(<\alpha)}$. We have the following induction hypotheses. Let $\tau \in T^{(\alpha)}$. So the sequence $\{M_{\tau \restriction \beta} \mid \beta < \alpha\}$ is defined. We assume that the sequence

$$(M_{\tau \restriction \beta}, Q_{\tau \restriction \beta}, f_{\tau \restriction \beta}), \quad \beta < \alpha$$

fulfills Clauses (H1)–(H4) from the proof of Theorem 2.7. More precisely, denote $M_{\tau \restriction \beta}$, $Q_{\tau \restriction \beta}$ and $f_{\tau \restriction \beta}$ by respectively $M^{\omega \cdot \beta}$, $Q^{\omega \cdot \beta}$ and $f^{\omega \cdot \beta}$. Then

$$\left(M^{\omega \cdot \beta}, Q^{\omega \cdot \beta}, f^{\omega \cdot \beta}\right), \quad \beta < \alpha$$

fulfills Clauses (H1)–(H4) from the proof of Theorem 2.7.

Let $\{S_\alpha \mid \alpha \in \omega_1\}$ be a $\diamondsuit$-sequence. We may assume that for every $\alpha \in \omega_1$, S_α is an object of the following form:

$$\langle \widehat{Q}_\alpha, \eta_\alpha, \zeta_\alpha, \hat{h}_\alpha, \widehat{I}_\alpha, \widehat{J}_\alpha \rangle,$$

where $\widehat{Q}_\alpha$, $\widehat{I}_\alpha$ and $\widehat{J}_\alpha$, are subsets of α, $\hat{h}_\alpha$ is a subset of $\alpha \times \alpha$, and η_α and ζ_α are subsets of $\alpha \times \{0,1\}$. (At stage α of the construction we shall use $S_{\omega \cdot \alpha}$.)

Recall that at stage α of the construction we have already constructed $\{M_\sigma \mid \sigma \in T^{(<\alpha)}\}$, and we construct $\{M_\sigma \mid \sigma \in T^{(\alpha)}\}$.

Case 1: $\alpha = 0$. Define M_Λ as M_ω was defined in the proof of Theorem 2.7.

Case 2: $\alpha \in \mathrm{Lim}(\omega_1)$. Let $\tau \in T^{(\alpha)}$. Set $\gamma = \omega \cdot \alpha$. Recall that the sequence $(M_{\tau \restriction \beta}, Q_{\tau \restriction \beta}, f_{\tau \restriction \beta})$, $\beta < \alpha$, fulfills Clauses (H1)–(H4) from the proof of Theorem 2.7. In this case we define M_τ, f_τ as in the limit case in the proof of Theorem 2.7, and we define Q_τ as in Theorem 2.7 using $\widehat{Q}_\gamma$. That is, define $P_\tau := f_\tau(\widehat{Q}_\gamma)$. If $P_\tau \subseteq I(B_{M_\tau})$, and is nowhere dense in M_τ, then let

$$Q_\tau = Q^{P_\tau}_{M_\tau} := \left\{\langle c, e\rangle \in \mathrm{Wip}(M_\tau) \;\middle|\; P_\tau \cap [c, -e] = \emptyset\right\}.$$

Otherwise, let $Q_\tau := \mathrm{Wip}(M_\tau)$.

Case 3: For some $\delta \in \mathrm{Lim}(\omega_1)$, $\alpha = \delta + 1$. Set $\gamma = \omega \cdot \alpha$. We distinguish between two subcases.

Case 3.1: Recall that $B_\sigma := B_{M_\sigma}$ and set $B^*_\sigma := B^*_{M_\sigma}$. Denote $\langle \widehat{Q}_\gamma, \eta_\gamma, \zeta_\gamma, \hat{h}_\gamma, \widehat{I}_\gamma, \widehat{J}_\gamma \rangle$ by $\langle \widehat{Q}, \eta, \zeta, \hat{h}, \widehat{I}, \widehat{J} \rangle$. Assume that:

(1) $\eta, \zeta \in \{0,1\}^\delta$, $\eta \neq \zeta$.
(2) $I := f_\eta(\widehat{I})$ is an ideal in B_η and for some $\beta < \delta$, $I \subseteq I_\beta(B_\eta)$.
(3) $J := f_\zeta(\widehat{J})$ is an ideal in B_ζ and for some $\beta < \delta$, $J \subseteq I_\beta(B_\zeta)$.
(4) Set $h = f_\zeta \circ \hat{h} \circ f_\eta^{-1}$. Then $h : B_\eta / I \cong B_\zeta / J$.

Case 3.2: Case 3.1 does not hold.

We start with the second case.

Case 3.2. Let $\tau \in \{0,1\}^\alpha$. Define M_τ, Q_τ and f_τ as in the successor case of Theorem 2.7. We give more details. For every $\beta \leqslant \delta$ define $M^{\omega \cdot \beta} = M_{\tau \restriction \beta}$, $Q^{\omega \cdot \beta} = Q_{\tau \restriction \beta}$ and $f^{\omega \cdot \beta} = f_{\tau \restriction \beta}$. Then the sequence

$$\left(M^{\omega \cdot \beta}, Q^{\omega \cdot \beta}, f^{\omega \cdot \beta}\right), \quad \beta \leqslant \delta$$

satisfies Clauses (H1)–(H4) from the proof of Theorem 2.7. Let $M^{\omega\cdot\alpha}$ and $f^{\omega\cdot\alpha}$ be as assured by the successor case of Theorem 2.7. We then set $M_\tau = M^{\omega\cdot\alpha}$ and $f_\tau = f^{\omega\cdot\alpha}$. Define $P_\tau := f_\tau(\widehat{Q}_{\omega\cdot\alpha})$. If $P_\tau \subseteq I(B_{M_\tau})$, and is nowhere dense in M_τ, then let

$$Q_\tau = Q^{P_\tau}_{M_\tau} := \bigl\{\langle c, e\rangle \in \mathrm{Wip}(M_\tau) \bigm| P_\tau \cap [c, -e] = \emptyset\bigr\}.$$

Otherwise, let $Q_\tau := \mathrm{Wip}(M_\tau)$.

Case 3.1. Let $\tau \in \{0,1\}^\alpha$. If $\tau \restriction \delta \notin \{\eta, \zeta\}$, then define M_τ, Q_τ and f_τ as in Case 3.2. It remains to define $M_{\eta\hat{}\langle\ell\rangle}$, $M_{\zeta\hat{}\langle\ell\rangle}$, etc. for $\ell = 0, 1$. In fact, we shall have that $M_{\eta\hat{}\langle 0\rangle} = M_{\eta\hat{}\langle 1\rangle}$ and $M_{\zeta\hat{}\langle 0\rangle} = M_{\zeta\hat{}\langle 1\rangle}$.

The sequence $\{(M_{\eta\restriction\beta}, Q_{\eta\restriction\beta}, f_{\eta\restriction\beta}) \mid \beta \leqslant \delta\}$ satisfies Clauses (H1)–(H4) from the proof of Theorem 2.7. Let $(M'_\eta, Q'_\eta, f'_\eta)$ be as in the successor case of Theorem 2.7. For ζ too, define $(M'_\zeta, Q'_\zeta, f'_\zeta)$ as in Theorem 2.7. Define

$$M_{\eta\hat{}\langle 0\rangle} = M_{\eta\hat{}\langle 1\rangle} = M'_\eta, \qquad Q_{\eta\hat{}\langle 0\rangle} = Q_{\eta\hat{}\langle 1\rangle} = Q'_\eta \quad \text{and} \quad f_{\eta\hat{}\langle 0\rangle} = f_{\eta\hat{}\langle 1\rangle} = f'_\eta.$$

Set $B'_\eta = B_{M'_\eta}$ and $B'_\zeta = B_{M'_\zeta}$.

Suppose first that there is no $g : B'_\eta/I \cong B'_\zeta/J$ such that $g \supseteq h$. Then define

$$M_{\zeta\hat{}\langle 0\rangle} = M_{\zeta\hat{}\langle 1\rangle} = M'_\zeta, \qquad Q_{\zeta\hat{}\langle 0\rangle} = Q_{\zeta\hat{}\langle 1\rangle} = Q'_\zeta \quad \text{and} \quad f_{\zeta\hat{}\langle 0\rangle} = f_{\zeta\hat{}\langle 1\rangle} = f'_\zeta.$$

Now assume that there is $g : B'_\eta/I \cong B'_\zeta/J$ such that $g \supseteq h$. Write $M_\zeta = \langle B^\star_\zeta, B_\zeta, D_\zeta\rangle$. Let $A \subseteq \widehat{\mathrm{At}}_\delta(B'_\zeta)$ be such that $B'_\zeta = \mathrm{cl}(B_\zeta \cup A)$, and for every distinct $a, b \in A$, $a \not\sim^{B'_\zeta} b$. Let $a_0 \in A$. Define $B = \mathrm{cl}(B_\zeta \cup (A \setminus \{a_0\}))$ and $B^* = \mathrm{cl}(B \cup D_\zeta)$. Let $N = \langle B^*, B, D_\zeta\rangle$. Then $M_\zeta \sqsubseteq N$. Since $B^* \subseteq B^*_{M'_\zeta}$, it follows that for every $\beta \leqslant \delta$, $\mathrm{Cnvnt}(Q_{\zeta\restriction\beta}, M_{\zeta\restriction\beta}; N)$ holds. We use the Successor Case Lemma 2.12. Let $d \in \mathcal{P}(\omega)$ be as assured by Lemma 2.12(b). Then by Lemma 2.12(b) and (c), $M_\zeta \sqsubseteq N[d]$, and for every $\beta \leqslant \delta$, $\mathrm{Cnvnt}(Q_{\zeta\restriction\beta}, M_{\zeta\restriction\beta}; N[d])$ holds. Define

$$M_{\zeta\hat{}\langle 0\rangle} = M_{\zeta\hat{}\langle 1\rangle} = N[d].$$

We then define $f_{\zeta\hat{}\langle 0\rangle}$, $Q_{\zeta\hat{}\langle 0\rangle}$, $f_{\zeta\hat{}\langle 1\rangle}$ and $Q_{\zeta\hat{}\langle 1\rangle}$ as in Case 3.2.

Case 4: $\alpha = 1$ or for some $\beta \in \omega_1$, $\alpha = \beta + 2$. For $\tau \in \{0,1\}^\alpha$ define M_τ, Q_τ and f_τ as in Case 3.2.

This concludes the definition of $\{M_\tau \mid \tau \in T\}$. Recall that for $\sigma \in \widehat{T}$, $M_\sigma = \bigcup_{\alpha<\omega_1} M_{\sigma\restriction\alpha}$ and $B_\sigma = B_{M_\sigma}$. For every $\sigma \in \widehat{T}$, M_σ is condensed. This is so, since for every $\sigma \in \widehat{T}$, the sequence $\{M_{\sigma\restriction\alpha} \mid \alpha < \omega_1\}$ fulfills Clauses (H1)–(H4) from the proof of Theorem 2.7.

We show that for every distinct $\sigma, \tau \in \widehat{T}$, $B_\sigma \not\simeq B_\tau$.

Claim 1. *Suppose that $M_{\eta\hat{}\langle j\rangle}$ and $M_{\zeta\hat{}\langle\ell\rangle}$ were obtained from the construction of Case 3.1. That is:*

(1) $\eta, \zeta \in \{0,1\}^\delta$, $\eta \neq \zeta$.
(2) *$I := f_\eta(\widehat{I})$ is an ideal in B_η and for some $\beta < \delta$, $I \subseteq I_\beta(B_\eta)$.*
(3) *$J := f_\zeta(\widehat{J})$ is an ideal in B_ζ and for some $\beta < \delta$, $J \subseteq I_\beta(B_\zeta)$.*
(4) *Set $h = f_\zeta \circ \hat{h} \circ f_\eta^{-1}$. Then $h : B_\eta/I \cong B_\zeta/J$.*

Set $\hat{\eta} = \eta\hat{}\langle j\rangle$ and $\hat{\zeta} = \zeta\hat{}\langle\ell\rangle$. Then there is no $f : B_{\hat{\eta}}/I \cong B_{\hat{\zeta}}/J$ such that $f \supseteq h$.

Proof. Let B'_η and B'_ζ be as in Case 3.1. If there is no $g : B'_\eta/I \cong B'_\zeta/J$ such that $g \supseteq h$, then Claim 1 is true since $B_{\hat{\eta}} = B'_\eta$ and $B_{\hat{\zeta}} = B'_\zeta$.

Now assume that $g : B'_\eta/I \cong B'_\zeta/J$ and $g \supseteq h$. Assume by contradiction that there is $f : B_{\hat{\eta}}/I \cong B_{\hat{\zeta}}/J$ such that $f \supseteq h$.

Let A, a_0 and B be as in Case 3.1. Then $B_{\hat{\zeta}} = B \subseteq B'_\zeta$. Hence $k := f \circ g^{-1}$ is an embedding of B'_ζ/J in itself. Also, since f, g extend h and $\mathrm{Dom}(h) = B_\zeta/J$, it follows that $k \restriction (B_\zeta/J) = \mathrm{Id}$. Since for some $\beta < \delta$, $J \subseteq I_\beta(B_\zeta)$, $\mathrm{At}(B'_\zeta/J) \subseteq B_\zeta/J$. Hence $k \restriction \mathrm{At}(B'_\zeta/J) = \mathrm{Id}$. So $k = \mathrm{Id}$. We show that $k(a_0/J) \neq a_0/J$. Clearly, $k(a_0/J) \in B/J$, for every $b \in a_0/J$, $b \sim^{B'_\zeta} a_0$ and there is no $b \in B$ such that $b \sim^{B'_\zeta} a_0$. So $k(a_0/J) \neq a_0/J$. A contradiction. $\square$

We prove the theorem. For every $\tau \in \widehat{T}$ the sequence $\{M_{\tau\restriction\alpha} \mid \alpha < \omega_1\}$ fulfills Clauses (H1)–(H4) from the proof of Theorem 2.7. Also, $M_\tau = \bigcup\{M_{\tau\restriction\alpha} \mid \alpha < \omega_1\}$. So M_τ is condensed.

Suppose by contradiction that there are distinct $\sigma, \tau \in \widehat{T}$, countable ideals $I \subseteq B_\sigma$, $J \subseteq B_\tau$, and h such that $h : B_\sigma/I \cong B_\tau/J$. Let $\gamma \in \omega_1$ be such that $\mathrm{rk}(I), \mathrm{rk}(J) < \gamma$ and $\sigma \restriction \gamma \neq \tau \restriction \gamma$. Let $\widehat{I} = f_\sigma^{-1}[I]$, $\widehat{J} = f_\tau^{-1}[J]$ and $\hat{h} = f_\tau^{-1} \circ h \circ f_\sigma$. Recall that

$$S_\alpha = \langle \widehat{Q}_\alpha, \eta_\alpha, \zeta_\alpha, \hat{h}_\alpha, \widehat{I}_\alpha, \widehat{J}_\alpha\rangle.$$

There is $\delta \in \omega_1$ such that $\gamma + \delta = \delta$, $\delta = \omega \cdot \delta$, $\eta_\delta = \sigma \upharpoonright \delta$, $\zeta_\delta = \tau \upharpoonright \delta$, $\widehat{I}_\delta = \widehat{I} \cap \delta$, $\widehat{J}_\delta = \widehat{J} \cap \delta$ and $\hat{h}_\delta = \hat{h} \upharpoonright \delta$. Then $M_{\sigma \upharpoonright (\delta+1)}$ and $M_{\tau \upharpoonright (\delta+1)}$ are constructed according to Case 3.1. By Claim 1,

($*$) there is no f such that $f : B_{\sigma \upharpoonright (\delta+1)}/I \cong B_{\tau \upharpoonright (\delta+1)}/J$ and $f \supseteq h \upharpoonright (B_{\sigma \upharpoonright \delta}/I)$.

Note that $I_{\delta+1}(B_\sigma/I) = I_{\delta+1}(B_\sigma)/I$. This follows from the fact that $\mathrm{rk}(I) + \delta = \delta$. The same holds for τ. Also, $I_{\delta+1}(B_\sigma) = I_{\delta+1}(B_{\sigma \upharpoonright (\delta+1)})$ and the same holds for τ. So we have

$$h\big[I_{\delta+1}(B_\sigma/I)\big] = I_{\delta+1}(B_\tau/J)$$

and

$$I_{\delta+1}(B_\sigma/I) = I_{\delta+1}(B_\sigma)/I = I_{\delta+1}(B_{\sigma \upharpoonright (\delta+1)})/I.$$

Hence

$$h\big[I_{\delta+1}(B_{\sigma \upharpoonright (\delta+1)})/I\big] = I_{\delta+1}(B_{\tau \upharpoonright (\delta+1)})/J.$$

Since $B_{\sigma \upharpoonright (\delta+1)}$ is generated by $I_{\delta+1}(B_{\sigma \upharpoonright (\delta+1)})$ and the same holds for τ,

$$h[B_{\sigma \upharpoonright (\delta+1)}/I] = B_{\tau \upharpoonright (\delta+1)}/J.$$

This means that

$$h \upharpoonright (B_{\sigma \upharpoonright (\delta+1)}/I) : B_{\sigma \upharpoonright (\delta+1)}/I \cong B_{\tau \upharpoonright (\delta+1)}/J.$$

But $h \upharpoonright (B_{\sigma \upharpoonright (\delta+1)}/I) \supseteq h \upharpoonright (B_{\sigma \upharpoonright \delta}/I)$. This contradicts ($*$). $\square$

3. Downward-categoricity of condensed Boolean algebras

The main goal of this section is the following theorem.

Theorem 3.1. *If B is a condensed Boolean algebra, then B is downward-categorical.*

Additional properties of condensed Boolean algebras are proved in Section 4.

Definition 3.2. (a) Let B be a Boolean algebra and I be a proper ideal in B. Then $\mathrm{B}_\pm(I)$ denotes $I \cup -I$. Recall that $\mathrm{B}_\pm(I)$ does not depend on B. Set $\mathrm{C}^B_\pm(I) := \mathrm{B}_\pm(\mathrm{cmpl}^B(I))$.

(b) Let B be a Boolean algebra. B is called a *rich Boolean algebra*, if it is thin–tall and unitary, and for every $\alpha < \omega_1$ there is a pure ideal I of B such that $\alpha \leqslant \mathrm{rk}(I) < \omega_1$.

The fact that a condensed BA is downward-categorical follows from the following two lemmas.

Lemma 3.3. *Let B be a rich Boolean algebra and J be a secluded ideal in B. Assume that* $\mathrm{rk}(J) < \omega_1$. *Let C be a subalgebra of B such that $C \supseteq \mathrm{C}^B_\pm(J)$. Then $C \cong B$.*

Lemma 3.4. *Let B be a condensed Boolean algebra, and C be an uncountable subalgebra of B. Then there is a pure ideal J in B such that* $\mathrm{rk}(J) < \omega_1$ *and $C \supseteq \mathrm{C}^B_\pm(J)$.*

Proof of Theorem 3.1 assuming Lemmas 3.3 and 3.4. The fact that B is a thin–tall algebra is trivial. Let C be an uncountable subalgebra of B. By Lemma 3.4, there is an ideal J of B such that J is pure, $\mathrm{rk}(J) < \omega_1$ and $C \supseteq \mathrm{C}^B_\pm(J)$. By the definitions of a condensed PI-system and of a PI-system (see (C2), (C3) and (P3)), B is rich. So by Lemma 3.3, C is isomorphic to B. $\square$

Definition 3.5. (a) For a countable ordinal α let $\mathbb{B}_\alpha$ denote the unique (up to isomorphism) unitary countable superatomic BA with rank α.

(b) For unitary BA's B_1 and B_2 we denote by $B_1 \odot B_2$ the subalgebra of $B_1 \times B_2$ generated by $I(B_1) \times \{0_{B_2}\} \cup \{0_{B_1}\} \times I(B_2)$.

Lemma 3.6.

(a) *Let B be a thin–tall Boolean algebra.*
  (1) *Let J be a pure ideal in B. Then $\mathrm{B}_\pm(J)$ is a unitary Boolean algebra, $\mathrm{C}^B_\pm(J)$ is a thin–tall Boolean algebra, and $B \cong \mathrm{C}^B_\pm(J) \odot \mathrm{B}_\pm(J)$.*

(2) *If $J \subseteq B$ is a pure ideal with rank β, then $\mathrm{B}_{\pm}(J) \cong \mathbb{B}_{\beta}$.*
(3) *Suppose that B is rich, and J is a secluded ideal in B such that $\mathrm{rk}(J) < \omega_1$. Then $\mathrm{C}^B_{\pm}(J) \cong B$.*

(b) *Let B be a rich BA, C be a subalgebra of B, and E be an ideal in B such that either E is principal generated by a member of $I(B)$, or E is a secluded ideal and $\mathrm{rk}(E) < \omega_1$. Suppose that $C \supseteq \mathrm{cmpl}^B(E)$. Then C is rich.*

Proof. (a1) It follows trivially from the definition of a pure ideal that $\mathrm{B}_{\pm}(J)$ is unitary and that $I(\mathrm{B}_{\pm}(J)) = J$. It is also trivial that $\mathrm{C}^B_{\pm}(J)$ is a thin–tall BA, and that $I(\mathrm{C}^B_{\pm}(J)) = \mathrm{cmpl}^B(J)$. So

$$\mathrm{C}^B_{\pm}(J) \odot \mathrm{B}_{\pm}(J) = \bigl\{\langle a, b\rangle \bigm| a \in \mathrm{cmpl}^B(J) \text{ and } b \in J\bigr\} \cup \bigl\{\langle -a, -b\rangle \bigm| a \in \mathrm{cmpl}^B(J) \text{ and } b \in J\bigr\}.$$

Let $f : \mathrm{C}^B_{\pm}(J) \odot \mathrm{B}_{\pm}(J) \to B$ be defined as follows:

$$f\bigl(\langle a, b\rangle\bigr) = \begin{cases} a + b & \text{if } a \in \mathrm{cmpl}^B(J) \text{ and } b \in J, \\ a \cdot b & \text{if } a \in -\mathrm{cmpl}^B(J) \text{ and } b \in -J. \end{cases}$$

It left to the reader to show that f is an isomorphism between $\mathrm{C}^B_{\pm}(J) \odot \mathrm{B}_{\pm}(J)$ and B.

(a2) The proof of (a2) is trivial.

(a3) It is trivial that $C := \mathrm{C}^B_{\pm}(J)$ is a thin–tall BA. Let I be a pure ideal of B such that $\mathrm{rk}(J) < \mathrm{rk}(I) < \omega_1$. Set $\beta = \mathrm{rk}(I)$. Let $I_1 := I \cap \mathrm{cmpl}^B(J)$. It is easy to see that I_1 is a pure ideal in both B and C, and that $\mathrm{rk}(I_1) = \beta$. Let J_1 be the ideal generated by $J \cup I_1$. By Proposition 2.2(e), J_1 is a pure ideal and $\mathrm{rk}(J_1) = \beta$. So by (a1) and (a2),

$$C \cong \mathrm{C}^C_{\pm}(I_1) \odot \mathrm{B}_{\pm}(I_1) \cong \mathrm{C}^C_{\pm}(I_1) \odot \mathbb{B}_{\beta} \quad \text{and} \quad B \cong \mathrm{C}^B_{\pm}(J_1) \odot \mathrm{B}_{\pm}(J_1) \cong \mathrm{C}^B_{\pm}(J_1) \odot \mathbb{B}_{\beta}.$$

A trivial computation shows that $\mathrm{C}^C_{\pm}(I_1) = \mathrm{C}^B_{\pm}(J_1)$. Hence that $B \cong C$. We make this computation. We just have to show that $\mathrm{cmpl}^C(I_1) = \mathrm{cmpl}^B(J_1)$. Indeed, $a \in \mathrm{cmpl}^C(I_1) \Leftrightarrow a \in C$ and $a \cdot I_1 = \{0\} \Leftrightarrow$ (i) $a \cdot J = \{0\}$ and $a \cdot I_1 = \{0\}$ *or* (ii) $-a \cdot J = \{0\}$ and $a \cdot I_1 = \{0\}$. However, since both I_1 and J are secluded ideals in B, there is no $a \in B$ satisfying (ii). So the above is equivalent to: (i) $a \cdot J = \{0\}$ and $a \cdot I_1 = \{0\}$. Clearly, (i) $\Leftrightarrow a \in \mathrm{cmpl}^B(I \cup J) \Leftrightarrow a \in \mathrm{cmpl}^B(I_1)$.

(b) Let I be a pure ideal in B such that $\mathrm{rk}(I) > \mathrm{rk}(E)$. Then $I \cap \mathrm{cmpl}^B(E)$ is a pure ideal in C and $\mathrm{rk}(I \cap \mathrm{cmpl}^B(E)) = \mathrm{rk}(I)$. This implies that C is rich. □

Proof of Lemma 3.3. Let B, J and C be as in Lemma 3.3. Let $C_1 := \mathrm{C}^B_{\pm}(J)$. By Lemma 3.6(a3), $C_1 \cong B$. We show that $C_1 \cong C$. The fact $C \supseteq \mathrm{cmpl}^B(J)$ and Lemma 3.6(b), imply that C is a rich. Since $C_1 \cong B$, C_1 is rich.

Let $J_1 := J \cap C$. Choose $J_2 \subseteq C_1$ such that J_2 is a pure ideal in C_1 and $\mathrm{rk}(J_2) > \mathrm{rk}(J_1)$.

Fact 1: J_2 is a pure ideal in C. We have that $J_2 \subseteq I(C_1)$ and that $I(C_1)$ is an ideal in C. So J_2 is an ideal in C. We now check properties (S1)–(S3) of Proposition 2.2(a). Obviously J_2 is non-principal and for every $\beta < \omega_1$, there is $b \in \mathrm{cmpl}^C(J_2)$ such that $\mathrm{rk}(b) = \beta$. So (S1) and (S3) hold. Let $c \in I(C)$. Write c as $c = c_1 + c_2$, where $c_1 \in J$ and $c_2 \in \mathrm{cmpl}^C(J)$. Then $c_2 \in C_1$. Moreover, $c_2 \in I(C_1)$. So there are $c_3 \in \mathrm{cmpl}^C(J_2)$ and $c_4 \in J_2$ such that $c_2 = c_3 + c_4$. Then $c = c_1 + c_3 + c_4$, $c_1 + c_3 \in \mathrm{cmpl}^C(J_2)$ and $c_4 \in J_2$. So (S2) holds.

Also, for every $a \in J_2$, $\mathrm{rk}^C(a) = \mathrm{rk}^{C_1}(a) < \mathrm{rk}(J)$. Hence J_2 is a pure ideal in C. □

Fact 2: Either J_1 is a principal ideal in C, or J_1 is a secluded ideal in C. Suppose that J_1 is not principal. Then (S1) holds. Clearly, $\mathrm{rk}(J_1) \leqslant \mathrm{rk}(J) < \omega_1 = \mathrm{rk}(C)$. So (S3) holds. Let $c \in I(C)$. Then $c \in I(B)$. Since J is either secluded or principal, there are $c_1 \in J$ and $c_2 \in \mathrm{cmpl}^B(J)$ such that $c = c_1 + c_2$. Since $C \supseteq \mathrm{cmpl}^B(J)$, it follows that $c_2 \in C$. Hence $c_1 \in C$. So $c_1 \in J_1$ and $c_2 \in \mathrm{cmpl}^C(J_1)$. That is, (S2) holds for J_1 and C. Hence J_1 is a secluded ideal in C. □

Let J_3 be the ideal of C generated by $J_1 \cup J_2$. Then by Proposition 2.2(e), J_3 is a pure ideal in C. By Lemma 3.6(a3),

$$\mathrm{C}^C_{\pm}(J_3) \cong C$$

and

$$\mathrm{C}^{C_1}_{\pm}(J_2) \cong C_1.$$

A trivial computation shows that $\mathrm{cmpl}^C(J_3) = \mathrm{cmpl}^{C_1}(J_2)$. So $\mathrm{C}^C_{\pm}(J_3) = \mathrm{C}^{C_1}(J_2)$. It follows that $C \cong C_1$. Since $C_1 \cong B$, it follows that $C \cong B$. □

To prove Lemma 3.4, we need another fact which will be used frequently.

Proposition 3.7.

(a) *Let M be any PI-system and $[c, -e]$ be a wide interval of M. If $e \notin I(B_M)$, then $[c, -e]^{B_M} \subseteq I(B_M)$. Also, every wide interval contains a wide interval $[c, -e]^{B_M}$ such that $e \notin I(B_M)$.*

(b) *Let B be a condensed Boolean algebra and M be a PI-system such that $B = B_M$. Any meet-closed uncountable subset P of $I(B)$ contains a wide interval.*

Proof. (a) The trivial proof is left to the reader.

(b) Let $[c_0, -e]^B$ be a wide interval such that P is dense in $[c_0, -e]^B$. By Part (a), we may assume that $e \notin I(B)$. Let $c \in P \cap [c_0, -e]^B$. Then P is dense in $[c, -e]^B$. We show that $[c, -e]^B \subseteq P$. Let $d \in [c, -e]^B$. By Part (a), $d \in I(B)$. So $[d, -e]^B$ is a wide interval. Choose $a \in P \cap (d, -e]^B$ and let $e' = e + (a - d)$. Then $\langle d, e' \rangle \in \mathrm{Wip}(M)$. Choose $b \in [d, -e']^B \cap P$. We have $a \cdot b = d$ and $a, b \in P$. Hence $d \in P$. That is $[c, -e]^B \subseteq P$. $\square$

Proof of Lemma 3.4. Let B be a condensed BA and M be a condensed PI-system such that $B = B_M$. Suppose that C is an uncountable subalgebra of B. So $C \cap I(B)$ is uncountable and meet-closed. By Proposition 3.7(b), there is $\langle c, e \rangle \in \mathrm{Wip}(M)$ such that $[c, -e] \subseteq C \cap I(B)$. Let $e_0 = c + e$ and $J_1 = B \restriction e_0$. We show that $C \supseteq \mathrm{cmpl}^B(J_1)$. Since $c \in [c, -e]$, we have that $c \in C$. Let $b \in \mathrm{cmpl}^B(J_1)$. Hence $b \leqslant -e$. So $c + b \in [c, -e] \subseteq C$. Also, $b \cdot c = 0$. We conclude that $b = (c + b) - c \in C$. That is, $C \supseteq \mathrm{cmpl}^B(J_1)$.

By Proposition 2.10(a) e_0 has the form $e_0 = (\sum D_0 - u) + v$, where D_0 is a finite subset of D_M and $u, v \in I(B)$. By the definition of a condensed PI-system, $\mathrm{rk}(B \restriction d) < \omega_1$ for every $d \in D_0$. So $\mathrm{rk}(J_1) < \omega_1$. It also follows from Proposition 2.10(a) that J_1 is secluded or that J_1 is principal. Let $d \in D_M$ be such that $\mathrm{rk}(B \restriction d) > \mathrm{rk}(J_1)$. Let $J_2 = B \restriction d$ and J be the ideal generated by $J_1 \cup J_2$. Then by Lemma 3.6(a3), J is pure and $\mathrm{rk}(J) < \omega_1$. Clearly, $\mathrm{cmpl}^B(J) \subseteq \mathrm{cmpl}^B(J_1)$. So $C \supseteq \mathrm{cmpl}^B(J)$. $\square$

We shall later see that a condensed Boolean algebra is never quotient-categorical. It is true however that, assuming CH, every uncountable quotient of a condensed BA is condensed. We need the following fact.

Corollary 3.8. *Let B be a condensed BA and $I \subseteq B$ be an uncountable ideal. Then $|B/I| \leqslant \aleph_0$.*

Proof. If $I \not\subseteq I(B)$, then it is trivial that $|B/I| \leqslant \aleph_0$. So suppose that $I \subseteq I(B)$. By Lemma 3.4, there is a countable pure ideal J such that $\mathrm{B}_{\pm}(I) \supseteq \mathrm{C}^B_{\pm}(J)$. Clearly, $\mathrm{cmpl}^B(J) \subseteq I(B)$ and $(-I) \cap I(B) = \emptyset$. So $I \supseteq \mathrm{cmpl}^B(J)$. It follows from the pureness of J that for every $a \in I(B)$ there is $a' \in J$ such that $a'/I = a/I$. So $|B/I| \leqslant |J| = \aleph_0$. $\square$

Theorem 3.9.

(a) *Let B be a narrow condensed BA, and I be an ideal of B such that B/I is uncountable. Then B/I is condensed.* (*See Remark* 2.8.)
(b) (CH) *Let B be a condensed BA and I be an ideal of B such that B/I is uncountable. Then B/I is condensed.*

Proof. We prove (a) and (b) together. Let $M = \langle B^\star, B, D \rangle$ be a condensed PI-system. By Corollary 3.8, $|I| \leqslant \aleph_0$, and hence $I \subseteq I(B)$. Let $\beta = \mathrm{rk}(I) \cdot \omega$. Set $T := \{d \in D \mid \mathrm{rk}(B \restriction d) \geqslant \beta\}$ and $S := D \setminus T$. Then $|S| \leqslant 2^{\aleph_0} = \aleph_1$. Let $\pi : S \to T$ be 1–1. For every $s \in S$ let $e_s = \pi(s) + s$ and set $E = \{e_s \mid s \in S\} \cup (T \setminus \mathrm{Rng}(\pi))$. Since $I \subseteq I(B)$, it follows that I is an ideal of $B^\star$. For every $b \in B^\star$, let b^I denote b/I, and for every subset $A \subseteq B^\star$, denote $A^I := \{a^I \mid a \in A\}$. Let $\widehat{B}$ be the subalgebra of $B^\star$ generated by $B \cup E$. Define $N = \langle \widehat{B}^I, B^I, E^I \rangle$.

We show that N is a condensed PI-system. Note first that for every $e \in E$, (A1) $\mathrm{rk}(B^I \restriction e^I) = \mathrm{rk}(B \restriction e)$ and (A2) $B \restriction e$ is a pure ideal in B.

We start with properties (P1)–(P4) of a PI-system. Clearly, B^I is unitary and $\widehat{B}^I$ is generated by $B^I \cup E^I$. By (A1), E^I is infinite. We show that B^I is dense in $\widehat{B}^I$. (This implies that $\widehat{B}^I$ is atomic and that $\mathrm{At}(\widehat{B}^I) = \mathrm{At}(B^I)$.)

Let $A = \{e - a \mid e \in E,\ a \in I(B)\}$. For every $y \in \widehat{B} \setminus B$: either (i) y is the finite sum of elements from A or (ii) $-y$ is the finite sum of elements from A. Suppose that (i) happens, and let $e - a \leqslant y$. Then there is $b \in I(B) \restriction (e - a)$ such that $\mathrm{rk}^B(b) > \beta$. So $0 \neq b^I \leqslant y^I$ and $b^I \in B^I$.

If (ii) happens, then $A \restriction y \neq \emptyset$. As in Case (i), it follows that $B^I \restriction y^I \neq \emptyset$.

We have shown that B^I is dense in $\widehat{B}^I$.

Let $e \in E$. By (A1), for every $a \in B^I \restriction e^I$, $\mathrm{rk}^{B^I}(a^I) < \mathrm{rk}(B^I \restriction e^I)$. So $B^I \restriction e^I$ is non-principal. Hence $e^I \notin B^I$. We have shown that (P1) and (P2) hold.

Now, for every $e \in E$, $B^I \restriction e^I$ is a pure ideal of B^I. This follows from the non-principality of $B^I \restriction e^I$ and from (A2). So (P3) holds.

For every distinct $e, f \in E$, $e \cdot f \in B$. So the same is true for e^I, f^I and B^I. So (P4) holds.

We have shown that N is a PI-system, and it remains to show that N fulfills properties (C1)–(C4) of a condensed PI-system. Properties (C1)–(C3) are trivial, and we prove (C4). Let $A \subseteq B$ be uncountable. We show that A^I is somewhere dense. Define $\rho : D \to E$ as follows: $\rho(d) = d + \pi(d)$ if $d \in S$, and $\rho(d) = d$ if $d \in T$. There is a wide interval $L := [c, -f]$ of M such that A is dense in L. And L has a wide subinterval of the form $[c_0, -(\sum D_0 - b)]$, where $c_0 \in I(B)$ and D_0 is a finite subset of D. Then $K := [c_0, -(\sum_{d \in D_0} \rho(d) - b - c_0)]$ is a wide subinterval of $[c_0, -(\sum D_0 - b)]$, and hence A is dense in K. The set $[c_0^I, -(\sum_{d \in D_0} \rho(d)^I - b^I - c_0^I)]$ is a wide interval of N and A^I is dense in this interval. $\square$

4. Rigidity and some other properties

A condensed BA is as rigid as a downward-categorical thin–tall BA could be. We explain this statement. A downward-categorical thin–tall BA must have countable secluded ideals. To see this, let B be a downward-categorical thin–tall BA. Let

$\alpha < \omega_1$ and $a \in \widehat{\mathrm{At}}_\alpha(B)$. Let A be the subalgebra of B generated by $I_\alpha(B \restriction a) \cup I(B \restriction -a)$. Since A is uncountable, $B \cong A$. Also, $I_\alpha(B \restriction a)$ is a pure ideal in A of rank α. This implies that B has pure ideals of unbounded rank.

Let B be a thin–tall BA and f be an endomorphism of B. We say that f is a *trivial endomorphism* if there is a countable secluded ideal I of B such that $f(a) = a$ for every $a \in \mathrm{cmpl}^B(I)$.

We consider a more general situation. Let B be a thin–tall BA, $I \subseteq B$ be an ideal and $f : B \to B/I$ be a homomorphism. We say that f is a *trivial homomorphism* if there is a countable secluded ideal J of B such that $f(a) = a/I$ for every $a \in \mathrm{cmpl}^B(J)$.

In this section we prove that if B is a condensed Boolean algebra, and f is an endomorphism of B such that $|\mathrm{Rng}(f)| = \aleph_1$, then f is a trivial endomorphism. In fact, we prove the analogous claim for homomorphisms.

Lemma 4.1. *Let B a condensed BA, $I \subseteq B$ be an ideal and $f : B \to B/I$ be a homomorphism such that $|\mathrm{Rng}(f)| = \aleph_1$. Let*

$$A = \bigl\{a \in B \bigm| \text{for every } b \in f(a),\ \mathrm{rk}(b \vartriangle a) \geqslant \mathrm{rk}(a)\bigr\}.$$

Then $|A| \leqslant \aleph_0$.

Proof. Let $B^\star$ and D be such that $M = \langle B^\star, B, D\rangle$ is a condensed PI-system. We show that for every $a \in I(B)$ and $b \in B$, if $f(a) = b/I$, then $b \in I(B)$. Suppose by contradiction that a and b refute the above. So

$$f[B \restriction -a] \subseteq (B/I) \restriction -f(a) = \{c/I \mid c \leqslant -b\}.$$

Since $B \restriction -b$ is countable, $f[B \restriction -a]$ is countable. Since $B \restriction a$ is countable, it follows that $f[B \restriction a]$ is countable. Hence $\mathrm{Rng}(f)$ is countable. A contradiction.

Since B/I is uncountable, I is countable. Let α be such that $I \subseteq I_\alpha(B)$. Set

$$C = \bigl\{a \in I(B) \bigm| \text{for every } b \in f(a),\ \mathrm{rk}(a - b) = \mathrm{rk}(a)\bigr\},$$
$$E = \bigl\{a \in I(B) \bigm| \text{for every } b \in f(a),\ \mathrm{rk}(b - a) \geqslant \mathrm{rk}(a)\bigr\}.$$

We show that $A \subseteq C \cup E \cup I_\alpha(B)$. Let $a \in A \setminus (C \cup E)$. We show that $\mathrm{rk}(a) < \alpha$.

There is $c \in f(a)$ such that $\mathrm{rk}(a - c) < \mathrm{rk}(a)$. There is $e \in f(a)$ such that $\mathrm{rk}(e - a) < \mathrm{rk}(a)$. Since $a \in A$, $\mathrm{rk}(c - a) \geqslant \mathrm{rk}(a)$,

$$c - e = c - (e \cap a) - (e - a).$$

Now, $c - (e \cap a) \supseteq c - a$. Hence $\mathrm{rk}(c - (e \cap a)) \geqslant \mathrm{rk}(c - a) \geqslant \mathrm{rk}(a)$. Recall that $\mathrm{rk}(e - a) < \mathrm{rk}(a)$. So

$$\mathrm{rk}(c - e) = \mathrm{rk}\bigl(\bigl(c - (e \cap a)\bigr) - (e - a)\bigr) = \mathrm{rk}\bigl(c - (e \cap a)\bigr) \geqslant \mathrm{rk}(a).$$

Since $c - e \in I$, $\mathrm{rk}(c - e) < \alpha$. Hence $\mathrm{rk}(a) < \alpha$. That is, $a \in I_\alpha(B)$. We have shown that $A \subseteq C \cup E \cup I_\alpha(B)$.

We leave it to the reader to show that $|C|, |E| \leqslant \aleph_0$. Since $A \subseteq C \cup E \cup I_\alpha(B)$, $|A| \leqslant \aleph_0$. $\square$

Proposition 4.2. *Let B a condensed BA, $I \subseteq B$ be an ideal and $f : B \to B/I$ be a homomorphism such that $|\mathrm{Rng}(f)| = \aleph_1$. Then $|\{a \in \widehat{\mathrm{At}}(B) \mid f(a) = a/I\}| = \aleph_1$.*

Proof. By Lemma 4.1, there is α such that:

- for every $a \in I(B) \setminus I_\alpha(B)$ there is $b_a \in f(a)$ such that $\mathrm{rk}(a \vartriangle b_a) < \mathrm{rk}(a)$.

For every countable $\beta \geqslant \alpha$ choose some $a_\beta \in \widehat{\mathrm{At}}_\beta(B)$. Denote b_{a_β} by b_β. By Fodor's Lemma, there is an uncountable set $A \subseteq \omega_1$ and $a \in B$ such that for every $\beta \in A$, $a_\beta \vartriangle b_\beta = a$. Let $\gamma = \min(A)$ and for every $\beta \in A \setminus \{\gamma\}$ let $c_\beta = a_\beta \vartriangle a_\gamma$ and $d_\beta = b_\beta \vartriangle b_\gamma$. Then $c_\beta \in \widehat{\mathrm{At}}_\beta(B)$ and $d_\beta \in f(c_\beta)$. Now,

$$c_\beta \vartriangle d_\beta = (a_\beta \vartriangle a_\gamma) \vartriangle (b_\beta \vartriangle b_\gamma) = (a_\beta \vartriangle b_\beta) \vartriangle (a_\gamma \vartriangle b_\gamma) = a \vartriangle a = 0.$$

That is, $c_\beta = d_\beta$. So $c_\beta \in f(c_\beta)$. $\square$

Theorem 4.3. *If B is a condensed Boolean algebra, $I \subseteq B$ is an ideal and $f : B \to B/I$ is a homomorphism such that $|\mathrm{Rng}(f)| = \aleph_1$, then f is a trivial homomorphism.*

Proof. The set $A := \{a \mid f(a) = a/I\}$ is a subalgebra of B. By Proposition 4.2, $|A| = \aleph_1$. Hence by Lemma 3.4, there is a countable secluded ideal I such that $A \supseteq \mathrm{C}^B_\pm(I) \supseteq \mathrm{cmpl}^B(I)$. So f is trivial. $\square$

Corollary 4.4. *Let B be a condensed Boolean algebra, I, J be ideals of B, and $f : B/I \to B/J$ be a homomorphism. Suppose that $|\mathrm{Rng}(f)| = \aleph_1$. Then there is a countable secluded ideal K of B such that $I \cap \mathrm{cmpl}(K) \subseteq J \cap \mathrm{cmpl}(K)$, and for every $a \in \mathrm{cmpl}(K)$, $f(a/I) = a/J$.*

Proof. Let π be the canonical homomorphism from B to B/I and $g = f \circ \pi$. By Theorem 4.3, there is a countable secluded ideal $K \subseteq B$ such that for every $a \in \mathrm{cmpl}(K)$, $g(a) = a/J$. Let $a \in I \cap \mathrm{cmpl}(K)$. Then

$$a/J = g(a) = f\big(\pi(a)\big) = f\big(0^{B/I}\big) = 0^{B/J}.$$

So $a \in J$. We have shown that $I \cap \mathrm{cmpl}(K) \subseteq J \cap \mathrm{cmpl}(K)$. Let $a \in \mathrm{cmpl}(K)$. Then

$$f(a/I) = f \circ \pi(a) = g(a) = a/J. \qquad \square$$

Corollary 4.5. *Let B be a condensed Boolean algebra, I be an ideal of B, and $f : B/I \to B$ be a homomorphism. Suppose that $|\mathrm{Rng}(f)| = \aleph_1$. Then there is a countable secluded ideal J of B such that*:

(1) $I \subseteq J$.
(2) *For every* $a \in \mathrm{cmpl}(J)$, $f(a/I) = a$.

Proof. Let $\pi : B \to B/I$ be the canonical homomorphism of B and I and $g = f \circ \pi$. Then g is an endomorphism of B and $|\mathrm{Rng}(g)| = \aleph_1$. So by Theorem 4.3, there is a secluded ideal J of B such that for every $a \in \mathrm{cmpl}(J)$, $g(a) = a$. Suppose by contradiction that $I \not\subseteq J$. Then since J is secluded, there is $a \in I \cap \mathrm{cmpl}(J) \setminus \{0\}$. Then $g(a) = 0 \neq a$. A contradiction, so $I \subseteq J$. Let $a \in \mathrm{cmpl}(J)$. Then $f(a/I) = g(a) = a$. $\square$

Recall that a Boolean algebra B is said to be quotient-categorical, if $|B| = \aleph_1$, and every uncountable quotient of B is isomorphic to B.

Definition 4.6. Let I be an ideal in a Boolean algebra B. A subalgebra C of B is a *retract* of I in B, if $I \cup C$ generates B and $C \cap I = \{0^B\}$. If every ideal of B has a retract, then B is said to be *retractive*. (Stated topologically, a Boolean algebra B is retractive iff every closed subset F of $\mathrm{Ult}(B)$ is a retract of $\mathrm{Ult}(B)$, that is, F is the range of a projection of $\mathrm{Ult}(B)$.)

We next show that a condensed Boolean algebra is never quotient-categorical. We also observe that condensed Boolean algebras are not retractive.

Corollary 4.7. *Let B be a condensed Boolean algebra.*

(a) *Let $I \subseteq B$ be an ideal. Then I has a retract iff either I is uncountable* (*and hence B/I is countable* (*Corollary* 3.8))*, or there is a countable secluded ideal J such that $I \subseteq J$.*
(b) *B is not retractive.*
(c) *B is not quotient-categorical.*

Proof. (a) We prove $\Rightarrow$. If I is uncountable, then B/I is countable. This implies that I has a retract. Suppose that J is a countable secluded ideal and $I \subseteq J$. The algebra $C := J \cup -J$ is countable, so $\langle C, I\rangle$ has some retract A_1. Let $A = \mathrm{cl}^B(A_1 \cup \mathrm{cmpl}(J))$. Then A is a retract for $\langle B, I\rangle$.

Proof of prove $\Leftarrow$. Let I be a countable ideal and A be a retract for I. There is a countable secluded ideal J such that $A \supseteq \mathrm{cmpl}(J)$. It follows that $I \subseteq J$.

(b) Since $I_1(B)$ is not contained in a countable secluded ideal, $I_1(B)$ does not have a retract.

(c) Let $C = B/I_1(B)$. We show that $C \not\cong B$. Suppose by contradiction that $f : C \cong B$. Define $g : B \to B$ as follows: $g(a) = f(a/I_1(B))$. Then g is an endomorphism from B onto B, and hence, $\mathrm{Rng}(g)$ is uncountable. It follows that g is a trivial endomorphism. That is, there is a countable secluded ideal I of B such that $g(a) = a$ for every $a \in \mathrm{cmpl}^B(I)$. Clearly, $\mathrm{At}(B) \cap \mathrm{cmpl}^B(I) \neq \emptyset$. Let $a \in \mathrm{At}(B) \cap \mathrm{cmpl}^B(I)$. Then $g(a) = a$. On the other hand, since $a \in I_1(B)$, it follows that $g(a) = 0$. A contradiction. $\square$

In [2] we defined the notion of a tightly Hausdorff space. In [3] (in preparation) we shall observe that a retractive space is tightly Hausdorff, and prove assuming $\diamondsuit_{\aleph_1}$, that there is a thin–tall space which is tightly Hausdorff but not retractive. We show here that the Stone space of a condensed Boolean algebra is not tightly Hausdorff.

Let X be a topological space and $x \in X$. Then $\mathrm{Nbr}^X(x)$ denotes the set of open neighborhoods of x. If $\mathcal{A}$ is a set of pairwise disjoint subsets of X, then

$$\mathrm{ac}^X(\mathcal{A}) := \big\{x \in X \bigm| \text{for every } U \in \mathrm{Nbr}(x),\ \{A \in \mathcal{A} \mid A \cap U \neq \emptyset\} \text{ is infinite}\big\}.$$

Definition 4.8. (a) Let X be a topological space and $A \subseteq X$. A family $\mathcal{U} := \{U_a \mid a \in A\}$ is a tight Hausdorff system for A, if for every $a \in A$, $U_a \in \mathrm{Nbr}(a)$, $\mathcal{U}$ is pairwise disjoint, and for every $B, C \subseteq \bigcup \mathcal{U}$: if

$$\{a \in A \mid B \cap U_a \neq \emptyset\} = \{a \in A \mid C \cap U_a \neq \emptyset\},$$

then

$$\operatorname{ac}\big(\{B \cap U_a \mid a \in U\}\big) = \operatorname{ac}\big(\{C \cap U_a \mid a \in U\}\big).$$

(b) A space X is *tightly Hausdorff space* if for every $A \subseteq X$: if A with its relative topology is a discrete space, then A has a tight Hausdorff system.

Corollary 4.9. *The Stone space of a condensed Boolean algebra is not tightly Hausdorff.*

Proof. Let B be a Boolean algebra, $A \subseteq B$ and $b \in B$. We say that b almost does not cut A, if $\{a \in A \mid a \cdot b, a - b \neq 0^B\}$ is finite. Let B be a superatomic Boolean algebra. Then the following are equivalent.

(1) $\operatorname{Ult}(B)$ is tightly Hausdorff.
(2) Let $A \subseteq \widehat{\operatorname{At}}(B)$ be such that for every $a \in A$ there is $a' \sim^B a$ such that for every $b \in A \setminus \{a\}$, $b - a' \sim^B b$. Then there is a pairwise disjoint family $C := \{c_a \mid a \in A\}$ such that for every $a \in A$, $c_a \sim^B a$ and such that for every $b \in B$, b almost does not cut C.

Let B be a condensed BA, and assume by contradiction that B is tightly Hausdorff. Let $A \subseteq \widehat{\operatorname{At}}_1(B)$ be such that for every $b \in \widehat{\operatorname{At}}_1(B)$ there is a unique $a \in A$ such that $a \sim^B b$. Then by (2), there is a pairwise disjoint family $C := \{c_a \mid a \in A\}$ such that for every $a \in A$, $c_a \sim^B a$ and such that for every $b \in B$ b almost does not cut C. Let

$$D = \big\{b \in B \,\big|\, \text{for every } c \in C,\ d \geqslant c \text{ or } d \cdot c = 0^B\big\}.$$

D is a subalgebra of B and D is uncountable. We show that D does not contain the complement of a countable secluded ideal. Let I be a countable secluded ideal. There is $c \in C$ such that $(\operatorname{At}(B) \restriction c) \cap I$ is finite. Let $e \in (\operatorname{At}(B) \restriction c) \setminus I$. Then $e \notin D$ and $e \in \operatorname{cmpl}(I)$. So $D \not\supseteq \operatorname{cmpl}(I)$. $\square$

We next observe that a condensed Boolean algebra cannot be well-generated. A Boolean algebra B is *well-generated*, if B has a sublattice L such that L generates B and $\langle L, <^B\rangle$ is well-founded. (See [1].)

Corollary 4.10. *If B is a condensed Boolean algebra, then B is not well-generated.*

Proof. We show that B does not have an uncountable well-founded sublattice. Let $A \subseteq B$ be uncountable. We show that the lower semi-lattice generated by A is not well-founded. Either $A \cap I(B)$ or $A \cap -I(B)$ are uncountable.

Suppose first that $A \cap -I(B)$ is uncountable. We may then assume that $A \subseteq -I(B)$. We define by induction $\{a_i \mid i \in \omega\} \subseteq A$. Choose $a_0 \in A$. Suppose that a_i has been defined for every $i < n$. The interval $[\prod_{i<n} a_i, 1^B]$ is a countable set. So there is $a_n \in A$ such that $a_n \notin [\prod_{i<n} a_i, 1^B]$. It follows that the sequence $\{\prod_{i\leqslant n} a_i \mid n \in \omega\}$ is a strictly decreasing sequence, and is contained in the lower semi-lattice generated by A.

Suppose next that $A \cap I(B)$ is uncountable. We may then assume that $A \subseteq I(B)$. Let $M = \langle B^\star, B, D\rangle$ be a condensed PI-system. By Proposition 3.7(b), A contains a wide interval of M. So A is not well-founded. $\square$

We have considered yet another topological property of sub-Ostaszewski Boolean algebras.

For a unitary algebra B set $\operatorname{Ult}^-(B) := \operatorname{Ult}(B) \setminus \{e^{\operatorname{Ult}(B)}\}$. One can easily verify that B is Ostaszewski iff B is sub-Ostaszewski, and every closed countable subset of $\operatorname{Ult}^-(B)$ is compact. So B is not Ostaszewski iff $\operatorname{Ult}^-(B)$ contains a countable closed non-compact set.

Definition 4.11. Let B be a sub-Ostaszewski algebra. B is *strongly non-Ostaszewski*, if it is not Ostaszewski, and for every closed countable subset $F \subseteq \operatorname{Ult}^-(B)$ there is a clopen countable subset $U \subseteq \operatorname{Ult}^-(B)$ such that $F \subseteq U$.

Under $\diamondsuit_{\aleph_1}$, it is easy to construct a sub-Ostaszewski algebra which is not strongly non-Ostaszewski.

Corollary 4.12. *A condensed Boolean algebra is strongly non-Ostaszewski.*

Proof. Let B be a condensed BA and $F \subseteq \operatorname{Ult}^-(B)$ be closed and countable. Let $I = \{a \in I(B) \mid \text{for every } x \in F,\ a \notin x\}$. Then I is an uncountable ideal. It follows easily from Lemma 3.4 that there is a countable secluded ideal J such that $I \supseteq \operatorname{cmpl}(J)$. Then U_J^B is a clopen subset of $\operatorname{Ult}^-(B)$ and $U_J^B \supseteq F$. $\square$

We mention one last property. Its easy proof is left to the reader.

Proposition 4.13. *If B is a condensed Boolean algebra, then every subset of B consisting of pairwise incomparable elements has cardinality $\leqslant \aleph_0$. In fact, more is true: For every uncountable set $P \subseteq I(B)$ there are distinct $a, b, c \in P$ such that $a \cdot b = c$. Note that this implies that every set of pairwise incomparable elements has cardinality $\leqslant \aleph_0$.*

5. The Ostaszewski case, packed Boolean algebras

Definition 5.1. (a) Let B be unitary. A *wide interval* of B is a set of the form $[c,-e]^B$, where $c, e \in I(B)$ and $c \cdot e = 0$. We call $\langle c, e\rangle$ a *wide interval pair* of B. Set $\mathrm{Wip}(B) := \{\langle c, e\rangle \in I(B) \times I(B) \mid c \cdot e = 0\}$.

(b) Let B be unitary and P be a subset of B. We say that P is *somewhere dense*, if there is $\langle c, e\rangle \in \mathrm{Wip}(B)$ such that for every $\langle c_1, e_1\rangle \in \mathrm{Wip}(B)$: if $[c_1, -e_1]^B \subseteq [c, -e]^B$, then $P \cap [c_1, -e_1]^B \neq \emptyset$.

(c) A Boolean algebra B is called a *packed Boolean algebra* if:

(1) B is thin–tall and unitary.
(2) Every uncountable subset of $I(B)$ is somewhere dense (and therefore every uncountable subset of B is somewhere dense).

We show that under $\diamondsuit_{\aleph_1}$ packed algebras exist, and that packed algebras are Ostaszewski algebras.

Theorem 5.2. *Assume $\diamondsuit_{\aleph_1}$. There is a packed Boolean algebra.*

Proof. In the definition of a PI-system remove the requirement that D is infinite, and in the definition of a condensed PI-system remove the requirement that for every $\alpha < \omega_1$ there is $d \in D_M$ such that $\mathrm{rk}(B_M \restriction d) \geqslant \alpha$. For a packed BA we need that D_M be the empty set.

The proof of the Construction Theorem 2.7 can be repeated, starting with a PI-system M_ω in which $D_\omega = \emptyset$, and omitting those parts in the proof in which D_M is enlarged. One obtains a (much simpler) PI-system M is which $B^\star_M = B_M := B$ and $D_M = \emptyset$. The argument which shows that every uncountable subset of $I(B)$ is somewhere dense remains the same. The resulting algebra B is packed. $\square$

Proposition 5.3. *Let B be a packed Boolean algebra.*

(a) *Let $I \subseteq B$ be an uncountable ideal. Then there is $a \in -I(B)$ such that $I \supseteq I(B) \restriction a$.*
(b) *B is an Ostaszewski algebra.*

Proof. (a) Any meet-closed uncountable subset P of $I(B)$ contains a wide interval. The proof of this fact is the same as in Proposition 3.7(b). Let I be an uncountable ideal. Then $I \cap I(B)$ is an uncountable ideal. Let $[c,-e]$ be a wide interval such that $I \cap I(B) \supseteq [c,-e]$. Then $I \cap I(B) \supseteq I(B) \restriction (-e-c)$ and $-e-c \in -I(B)$.

(b) We prove that B is sub-Ostaszewski. By the remark following Definition 1.7, it suffices to show that if $I \subseteq B$ is an uncountable ideal, then B/I is countable. Let I be such an ideal. By Part (a), there is $a \in -I(B)$ such that $I \supseteq I(B) \restriction a$. Then $\{b/I \mid b \in B \restriction (-a)\} = B/I$. So B/I is countable.

To show that B is Ostaszewski it remains to prove that every countable closed subset of $\mathrm{Ult}^-(B)$ is compact. Let F be such a set. Define $I = \{a \in I(B) \mid \text{for every } x \in F,\ a \notin x\}$. Note that $U^B_I = \mathrm{Ult}^-(B) \setminus F$. Hence I is uncountable. Let $b \in -I(B)$ be such that $I(B) \restriction b \subseteq I$. Set $J = B \restriction -b$. We show that $F \subseteq U^B_J$. Let $x \in F$. Suppose by contradiction that $b \in x$. Let $c \in x \cap I(B)$. Then $c \cdot b \in I(B) \restriction b$ and hence $c \cdot b \in I$. However, $c \cdot b \in x$. This contradicts the definition of I. So $b \notin x$. Hence $-b \in x$. So $x \in U^B_J$. U^B_J is compact, $U^B_J \subseteq \mathrm{Ult}^-(B)$ and $F \subseteq U^B_J$. So F is compact. $\square$

As mentioned, an Ostaszewki algebra cannot be downward-categorical. In particular, packed Boolean algebras are not downward-categorical. However, all the other properties of a condensed BA, those which are discussed in Section 4, have a counterpart in the "packed" case. In essence, if in the condensed case we say that something happens "outside a countable secluded ideal", then in the "packed" case we obtain that the same thing happens "outside a countable principal ideal". In particular, we have the following conclusion.

Corollary 5.4. *Let B be packed. Then B is rigid in the following sense. Suppose that I, J are ideals in B and $f : B/I \to B/J$ is a homomorphism with an uncountable range. Then there is a principal ideal K of B such that $|\mathrm{cmpl}(K)| \leqslant \aleph_0$, $I \cap K \subseteq J \cap K$, and for every $a \in K$, $f(a/I) = a/J$.*

Remark 5.5. The following variation of Corollary 5.4 can be proved assuming only (CH). There is a thin–tall BA B such that the following holds. Let I, J be ideals in B with $|I|, |J| \leqslant \aleph_0$, and let $f : B/I \to B/J$ be a homomorphism. *Then* there is a principal ideal K of B such that $|\mathrm{cmpl}(K)| \leqslant \aleph_0$, $I \cap K \subseteq J \cap K$, and for every $a \in K$, $f(a/I) = a/J$.

In fact, there are $2^{\aleph_1}$ pairwise non-isomorphic such algebras.

6. An observation and a question

Propositions 6.1 and 6.2 were found by M. Weese [7] 1991.

Proposition 6.1. *Let B be a downward-categorical algebra. Then B is superatomic.*

Proof. Suppose that B is not superatomic. So B contains a chain isomorphic to $\mathbb{Q}$. Let $b \in B$ be such that $B \restriction b$ contains a chain isomorphic to $\mathbb{Q}$ and $B \restriction -b$ is uncountable. For each countable ordinal α let C_α be a subalgebra of $B \restriction b$ isomorphic to $\mathbb{B}_\alpha$. So $C_\alpha \times (B \restriction -b)$ is uncountable and thus isomorphic to B. For every $\alpha < \omega_1$ let $a_\alpha \in B$ be such that $B \restriction a_\alpha \cong \mathbb{B}_\alpha$, and let C be the subalgebra of B generated by $\{a_\alpha \mid \alpha < \omega_1\}$. Then C is superatomic and $C \cong B$. A contradiction. $\square$

Proposition 6.2. *Assume $2^{\aleph_0} < 2^{\aleph_1}$. Let B be a downward-categorical algebra. Then either B is thin–tall or B is isomorphic to the algebra of finite and cofinite subsets of ω_1.*

Proof. Suppose that B is not thin–tall. Clearly B is unitary and $\mathrm{rk}(B) \leqslant \omega_1$. Let $\alpha < \omega_1$ be the first ordinal such that $\widehat{\mathrm{At}}_\alpha(B)$ is uncountable. Assume by contradiction that $\alpha > 0$. So $\mathrm{At}(B)$ and $I_\alpha(B)$ are countably infinite. Let C be the subalgebra generated by $I_\alpha(B) \cup \widehat{\mathrm{At}}_\alpha(B)$. Since C is uncountable, it is isomorphic to B.

For each uncountable subset A of $\widehat{\mathrm{At}}_\alpha(B)$, we denote by $B(A)$ the subalgebra of B generated by $I_\alpha(B) \cup A$. Clearly, $|\{B(A) \mid A \subseteq \widehat{\mathrm{At}}_\alpha(B)\}| = 2^{\aleph_1}$. That is, B has $2^{\aleph_1}$ distinct uncountable subalgebras.

It is trivial that if f, g are embeddings of B into B and $f \restriction \mathrm{At}(B) = g \restriction \mathrm{At}(B)$, then $f = g$. So B has at most $2^{\aleph_0}$ embeddings into itself. A contradiction.

It follows that $\alpha = 0$. Hence $|\mathrm{At}(B)| > \aleph_0$. This implies that the Boolean algebra F of finite and cofinite subsets of ω_1 is embeddable in B. So $B \cong F$. $\square$

We conclude with a question.

Question 6.3. (a) Is it consistent that there is a thin–tall downward-categorical BA B such that B has an automorphism f such that for every $\alpha < \omega_1$ there is $a \in \widehat{\mathrm{At}}_\alpha(B)$ such that $f(a) \cdot a = 0$?

(b) Is it consistent that there is a thin–tall downward-categorical BA B such that B has an automorphism f such that for every for every $a \in \mathrm{At}(B)$, $f(a) \neq a$?

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References

[1] R. Bonnet, M. Rubin, On well-generated Boolean algebras, Ann. Pure Appl. Logic 105 (1–3) (2000) 1–50.
[2] R. Bonnet, M. Rubin, A classification of CO spaces which are continuous images of compact ordered spaces, Topology Appl. 155 (5) (2008) 375–411.
[3] R. Bonnet, M. Rubin, A thin–tall space which is tightly Hausdorff but not retractive, 2011, in preparation.
[4] R. Bonnet, S. Shelah, On HCO spaces. An uncountable compact T_2 space, different from $\aleph_1 + 1$, which is homeomorphic to each of its uncountable closed subspaces, Israel J. Math. 84 (3) (1993) 289–332.
[5] A. Dow, P. Simon, Thin–tall Boolean algebras and their automorphism groups, Algebra Universalis 29 (2) (1992) 211–226.
[6] T. Eisworth, J. Roitman, CH with no Ostaszewski spaces, Trans. Amer. Math. Soc. 351 (7) (1999) 2675–2693.
[7] J.D. Monk, M. Weese, Subalgebra-rigid and homomorphism-rigid Boolean algebras, 1991, manuscript.
[8] J. Roitman, A space homeomorphic to each uncountable closed subspace under CH, Topology Appl. 55 (1994) 273–287.
[9] J. Roitman, Maps of Ostaszewski and related spaces, Topology Appl. 72 (1996) 121–133.
[10] J. Roitman, More homogeneous almost disjoint families, Algebra Universalis 47 (2002) 267–282.
[11] M. Rubin, A Boolean algebra with few subalgebras, interval Boolean algebras and retractiveness, Trans. Amer. Math. Soc. 278 (1) (1983) 65–89.
[12] M. Weese, On the classification of superatomic Boolean algebras, in: Open Days in Model Theory and Set Theory, Proc. Conf. Jadwissin, University of Leeds, 1986.